On Some Matching Problems Under the Color-Spanning Model

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Abstract

Given a set of n points Q in the plane, each colored with one of the k given colors, a color-spanning set $S \subset Q$ is a subset of k points with distinct colors. The minimum diameter color-spanning set (MDCS) is a color-spanning set whose diameter is minimum. Somehow symmetrically, the largest closest pair color-spanning set (LCPCS) is a color-spanning set whose closest pair is the largest. Both MDCS and LCPCS have been shown to be NP-complete, but whether they are fixed-parameter tractable (FPT) when k is a parameter is open. Motivated by this question, we consider the FPT tractability of some matching problems under this color-spanning model, where $2k$ is the parameter. We show that the following three problems are polynomially solvable (hence FPT): (1) MinSum Matching Color-Spanning Set, (2) MaxMin Matching Color-Spanning Set, and (3) MinMax Matching Color-Spanning Set. For the k-Multicolored Independent Matching problem, namely, computing a matching of $2k$ vertices in a graph such that the vertices of the edges in the matching do not share edges, we show that it is W[1]-hard. Finally, motivated by this problem, which is related to the parameterized independent set problem, we are able to prove that LCPCS is W[1]-hard.

Keywords: (Fixed-parameter) Computational geometry, Matching

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algorithms, FPT algorithms, Color-spanning model, NP-completeness

1. Introduction

Given a set of n points P with all points colored in one of the t given colors, a *color-spanning* set (sometimes also called a *rainbow* set) is a subset of t points with distinct colors. (In this paper, as we focus on matching problems on P, we set $t = 2k$ henceforth. For other cases, we use a point set Q and the parameter k, which does not have to be even.) In practice, many problems require us to find a specific color-spanning set with certain property due to the large size of the color-spanning sets. For instance, in data mining a problem arises where, given a set Q of n points colored in k colors, one wants to find a color-spanning set whose diameter is minimized, which can be solved in $O(n^k)$ time using a brute-force method [20, 3]. (Unfortunately, this is still the best bound to this date.)

Since the color-spanning set problems were initiated in 2001 [1], quite some related problems have been investigated. Many of the traditional problems which are polynomially solvable, like Minimum Spanning Tree, Diameter, Closest Pair, Convex Hull, etc, become NP-hard under the color-spanning model [8, 9, 13]. Note that for the hardness results the objective functions are usually slightly changed. For instance, in the color-spanning model, we would like to maximize the closest pair and minimize the diameter (among all color-spanning sets). On the other hand, some problems, like Maximum Diameter Color-Spanning Set, remain to be polynomially solvable [6].

In [8, 9], an interesting question was raised. Namely, if t is a parameter, is the NP-complete Minimum Diameter Color-Spanning Set (MDCS) problem fixed-parameter tractable? This question is still open. In this paper, we try to investigate some related questions along this line. The base problem we target at is the matching problem, both under the geometric model and the graph model. We show that an important graph version is W[1]-hard while all other versions in consideration are polynomially solvable, hence are fixed-parameter tractable (FPT). Motivated by this problem, which is somehow related to the k-independent set problem, we show that Largest Closest Pair Color-Spanning Set (LCPCS) is W[1]-hard.

This paper is organized as follows. In Section 2, we define the basics regarding FPT algorithms and the problems we will investigate. In Section 3, we show that MinSum, MaxMin and MinMax Matching Color-Spanning Set are all polynomially solvable by reducing to the minimum weight matching problem. In Section 4, we show that a special graph version is W[1]-hard and, on top of that, we show that LCPCS is W[1]-hard. In Section 5, we conclude the paper.

2. Preliminaries

We make the following definitions regarding this paper. A Fixed-Parameter Tractable (FPT) algorithm is an algorithm for a decision problem with input size n and parameter k whose running time is $O(f(k)n^c) = O^*(f(k))$, where $f(-)$ is any computable function on k and c is a constant. FPT algorithms are efficient tools for handling some NP-complete problems as they introduce an extra dimension k. If an NP-complete problem, like Vertex Cover, admits an FPT algorithm, then it is basically polynomially solvable when the parameter k is a small constant [5, 10].

Of course, it is well conceived that not all NP-hard problems admit FPT algorithms. It has been established that

$$\text{FPT} \subseteq W[1] \subseteq W[2] \subseteq \cdots W[z] \subseteq \text{XP},$$

where XP represents the set of problem which must take $O(n^k)$ time to solve (i.e., not FPT), with k being the parameter. Typical problems in W[1] include Independent Set and Clique, etc. For the formal definition and foundation, readers are referred to [5, 10].

Given a set Q of n points in the plane with k colors, a *color-spanning* set $S \subset Q$ is a subset of k points with distinct colors. If S satisfies a property Π among all color-spanning sets of Q, we call the corresponding problem of computing S the Property-Π Color-Spanning Set. For instance, the Minimum Diameter Color-Spanning Set (MDCS) is one where the diameter of S is minimized (among all color-spanning sets of Q) and the Largest Close Pair Color-Spanning Set (LCPCS) is one where the closest pair of S is maximized (among all color-spanning sets of Q). The distance measure for two points in the plane is the Euclidean (or L_2) distance. We next define the matching problems we will investigate in this paper.

Given a set P of n points in the plane with $t = 2k$ colors, let $S \subset P$ be a color-spanning set of $2k$ distinct colors. Then the (disjoint pairs of) points in S always *induce* a perfect matching, i.e., a set M of k edges connecting the $2k$ points in S. Among all these matchings (over all color-spanning sets), if a matching M satisfies a property Π, we call the problem the *Property-*Π *Matching Color-Spanning Set* or *Property-*Π *Color-Spanning Matching*. The three properties we focus on are MinSum, MinMax and MaxMin.

MinSum means that the sum of edge lengths in M is minimized, MinMax means that the maximum edge length in M is minimized, and MaxMin means that the minimum edge length in M is maximized. One of the the main purposes of this paper is to investigate the FPT tractability of the three problems: MinSum Matching Color-Spanning Set, MinMax Matching Color-Spanning Set, and MaxMin Matching Color-Spanning Set. We show that all these problems are in fact polynomially solvable (hence FPT).

Finally, we will study a special version on graphs where the (vertices of the) edges in M cannot share edges in G. We call the problem k-Multicolored Independent Matching, and we will show that this problem is W[1]-hard. This problem eventually helps us prove that Largest Closest Pair Color-Spanning Set (LCPCS) is W[1]-hard. In the next section, we first show the positive results. The negative W[1]-hardness results will be shown in Section 4.

3. MinSum, MaxMin and MinMax Matching Color-Spanning Set Problems

3.1. MinSum and MaxMin Matching Color-Spanning Set are in P

We first consider the *MinSum Matching Color-Spanning Set* problem. Formally, given a set P of n points in the plane, each colored with one of the $2k$ colors, we need to identify $2k$ points with distinct colors such that they induce a matching with certain property (e.g., the minimum total weight). Recall that the weight of an edge (p_i, p_j) is the Euclidean distance between p_i and p_j. For a point p_i, let $color(p_i)$ be the color of p_i.

For *MaxMin Matching Color-Spanning Set*, the first attempt is to try to see its relation to the MinSum Matching Color-Spanning Set problem. In Figure 1, we show an example where MaxMin Matching Color-Spanning Set is not necessarily related to the MinSum (or MaxSum) Color-Spanning Matching. In Figure 1, the MinSum Color-Spanning Matching is $\{(a, c), (b, f)\}$, with a total weight of $2 - 2\epsilon$. The MaxSum Color-Spanning Matching is $\{(a, b), (d, e)\}$, which has a total weight of $1 + \sqrt{5}$. The optimal solution for MaxMin Color-Spanning Matching is $\{(a, d), (b, e)\}$, with a solution value of $\sqrt{2}$ (while the total weight is $2\sqrt{2}$). Note that (a, d) and (b, e) do not form the closest pairs among the subsets of respective colors.

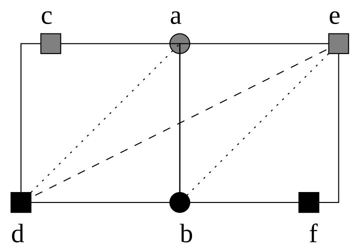

Figure 1. An example of a 4-colored set of 6 points in the plane. The edges of both squares have length 1. Points c, f are ϵ distance away from the corresponding closest square corners. The MinSum Color-Spanning Matching is $\{(a, c), (b, f)\}$, with a minimum edge weight of $1 - \epsilon$. The MaxMin Color-Spanning Matching is $\{(a, d), (b, e)\}$, with a minimum edge weight of $\sqrt{2}$.

For the same point set $\{a, b, c, d, e, f\}$, the color-spanning set $\{a, b, d, e\}$ (which happens to correspond to the point set for MaxMin Color-Spanning Matching), gives the solution for LCPCS (largest closest pair color-spanning set). The corresponding closest pair in the set has length 1, while the solution value for MaxMin Color-Spanning Matching is $\sqrt{2}$. Hence, LCPCS and MaxMin Color-Spanning Matching are not the same and the claim we made in the conference version [2] is not correct. In fact, as we will see a bit later, not only that the two problems are not the same, they are quite different: the former problem is W[1]-hard while the latter is polynomially solvable (hence FPT).

Nonetheless, we show next that MinSum and MaxMin Matching Color-Spanning Set have the following property.

Lemma 1. *In an optimal solution of MinSum (resp. MaxMin) Matching Color-Spanning, let p_i and p_j form a (resp. minimum) matched edge in the optimal matching, then (p_i, p_j) must be the closest (resp. farthest) pair between points of $color(p_i)$ and $color(p_j)$.*

Proof. The proof for the two cases are almost identical, so we only consider the maxmin case. Let $d_1(p_i, p_j)$ be the length of the minimum matched edge. Let $d_2(p_i, p_j)$ be the length of the farthest pair between points of $color(p_i)$ and $color(p_j)$. Then we could replace $d_1(p_i, p_j)$ by $d_2(p_i, p_j)$ to have a new matching whose minimum matched edge length is longer. $\square$

Using this property, we show that MinSum Matching Color-Spanning Set can be solved in polynomial time (hence FPT). First, for all $\binom{2k}{2}$ pairs of colors, compute the bichromatic closest pair of points of the selected colors. This can be done in $O(n \log n)$ time [19] for each pair of colors. The total time for all pairs of colors is $O(k^2 n \log n)$. It can be reduced to $O(kn \log n)$ as follows. Suppose that the colors are $1, 2, \ldots, 2k$. For each $i = 1, 2, \ldots, 2k-1$, do the following steps.

(1) Make a graph $G = (V, E)$ with $V = \{1, 2, \ldots, 2k\}$ and $E = \emptyset$.
(2) For points of color i, construct the Voronoi diagram and a data structure D_i for point location with $O(\log n)$ query time.
(3) For each color $j \in \{i+1, i+2, \ldots, 2k\}$ and each point p of color j, find its nearest neighbor q in D_i. For each color $j \in \{i+1, i+2, \ldots, 2k\}$, compute a pair (p, q) with minimum Euclidean distance and add it to E.

Finally, we compute a perfect matching in G of minimum weight using a variation of Edmonds algorithm with running time $O(n^3)$, where n is the number of vertices of G [14, 11][1]. We hence have the following theorem.

Theorem 1. *A minsum matching color-spanning set can be computed in* $O(k^3 + kn \log n)$ *time.*

Similarly, we show that MaxMin Matching Color-Spanning Set is polynomially solvable (hence FPT). With Lemma 1, we construct a complete graph G_1 over k vertices each corresponding to one of the k colors and between two colors c_i, c_j we have an edge whose weight (length) $w(c_i, c_j)$ is the farthest pair (distance) between points of color c_i and c_j. The cost for constructing G_1 is $O(kn \log n)$ time.

To solve the problem, we sort all edges in G_1. Then for any given edge $e = (c_i, c_j) \in E(G_1)$, we delete all edges of lengths smaller than $w(e)$ and we delete c_i, c_j as well from G_1. Let G_1' be the resulting graph (containing $2k-2$ colors). Then the problem is to test whether G_1' contains a perfect matching saturating the remaining $2k-2$ colors. The total cost for this decision problem is $O(k^3)$ [14, 11]. We then could use binary search to find the best e^* in $O(k^3 \log k)$ time. The total cost of this algorithm is $O(k^3 \log k + kn \log n)$ time.

[1]Notice that the problems of finding the matchings of minimum weight and of maximum weight are equivalent.

Corollary 1. *MaxMin Matching Color-Spanning Set can be solved in $O(k^3 \log k + kn \log n)$ time.*

3.2. MinMax Matching Color-Spanning Set is in P

In this subsection, we consider MinMax Matching Color-Spanning Set. Not surprisingly, such a matching has nothing to do with the MinSum Color-Spanning Matching or the MaxSum Color-Spanning Matching. In Figure 2, the MinSum Color-Spanning Matching is $\{(a, b), (c, d)\}$, with a total weight of 3. The MaxSum Color-Spanning Matching has a weight at least that of $\{(a, c), (b, d)\}$ or $\{(c, d), (e, f)\}$, each having a total weight of $4 + 2\epsilon$. For the MinMax Color-Spanning Matching problem, all of the above matchings give a solution value of $2 + \epsilon$. The optimal solution is $\{(c, e), (d, f)\}$, with a solution value of $1.5 + \epsilon$ (while the total weight is $3 + 2\epsilon$). Also, note that (c, e) and (d, f) do not form the farthest pairs among the subsets of respective colors.

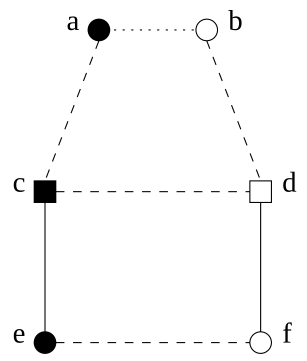

Figure 2. A simple multicolored point set, the dotted, dashed and solid segments have lengths $1 - \epsilon$, $2 + \epsilon$ and $1.5 + \epsilon$ respectively. The MinSum color-spanning matching is $\{(a, b), (c, d)\}$, with a maximum edge weight of $2 + \epsilon$. The MinMax color-spanning matching is $\{(c, e), (d, f)\}$, with a maximum edge weight of $1.5 + \epsilon$.

We next state that MinMax Matching Color-Spanning Set has the following property, which is a corollary of Lemma 1.

Corollary 2. *In an optimal solution of MinMax Color-Spanning Matching, let p_i and p_j be the maximum matched edge, then (p_i, p_j) must be the closest pair between points of color(p_i) and color(p_j).*

We could solve MinMax Color-Spanning Matching in very much the same way as in Corollary 1, in $O(k^3 \log k + kn \log n)$ time. However, after a graph G_2, over $2k$ colors and the edge weights between two colors being the closest pair between the corresponding colors, is constructed, we note that the problem is really the Bottleneck Matching problem on G_2. For a graph with n_V vertices and n_E edges, it is known that such a matching can be computed in $O(\sqrt{n_V \log n_V} \cdot n_E)$ time [12]. Hence, in our case the MinMax Color-Spanning Matching can be solved in $O(k^{2.5}\sqrt{\log k} + kn \log n)$ time. Therefore, we have the following corollary.

Corollary 3. *MinMax Color-Spanning Matching can be solved in* $O(k^{2.5}\sqrt{\log k} + kn \log n)$ *time.*

In the next section, we show that a special matching problem on graphs is in fact W[1]-hard. This helps us find some ideas to prove that Largest Closest Pair Color-Spanning Set is W[1]-hard.

4. *k*-Multicolored Independent Matching is W[1]-hard

The k-Multicolored Independent Matching problem is defined as follows.

INSTANCE: An undirected graph $G = (V, E)$ with each vertex colored with one of the $2k$ given colors.

QUESTION: Is there an independent matching $E' \subseteq E$ including all the k colors? That is, are there k edges in E' such that all the vertices of the edges in E' have different colors, and for any two edges (x_1, x_2) and (y_1, y_2) in E', $(x_i, y_j) \notin E$ (with $i, j = 1..2$).

The problem originates from an application in shortwave radio broadcast, where the matched nodes represent the shortwave channels which should not directly affect each other [15]. (We also comment that this problem seems to be related to the uncolored version of 'Induced Matching' which is known to be W[1]-hard as well [17, 18].) We will show that this problem is not only NP-complete but also W[1]-hard. The problem to reduce from is the k-Multicolored Independent Set, which is defined as follows.

INSTANCE: An undirected graph $G = (V, E)$ with each vertex colored with one of the k given colors.

QUESTION: Is there an independent set $V' \subseteq V$ including all the k colors? That is, are there k vertices in V' incurring no edge in E, and all the vertices in V' have different colors.

When $U \subseteq V$ contains exactly k vertices of different colors, we also say that U is *colorful*.

For completeness, we first prove the following lemma, similar to what was done by Fellows *et al.* on k-Multicolored Clique problem [7].

Lemma 2. *k-Multicolored Independent Set is W[1]-hard.*

Proof. The proof can be done through a reduction from k-Independent Set. Let $G = (V, E)$ be a general connected graph. Given an instance $(G = (V, E), k)$ for k-Independent Set, we first make k copies of G, G_i's, such that the vertices in each G_i are all colored with color i, for $i = 1..k$. For any $u \in V$, let u_i be the corresponding mirror vertex in G_i. Then, for each $(u, v) \in E$ and for each pair of i, j, with $1 \leq i \neq j \leq k$, we add four edges (u_i, u_j), (v_i, v_j), (u_i, v_j) and (u_j, v_i). Let the resulting graph be G'. It is easy to verify that G has a k-independent set if and only if G' has a k-multicolored independent set. As k-Independent Set is W[1]-complete [5], the lemma follows. □

The following theorem shows that k-Multicolored Independent Matching is not only NP-complete but also W[1]-hard.

Theorem 2. *k-Multicolored Independent Matching is W[1]-hard.*

Proof. We reduce k-Multicolored Independent Set (IS) to the k-Multicolored Independent Matching problem.

Given an instance of k-Multicolored IS problem, i.e., a connected graph $G = (V, E)$ with each vertex in $V = \{v_1, v_2, ..., v_n\}$ colored with one of the k colors $\{1, 2, ..., k\}$, the question is whether one could compute an IS of size k, each with a distinct color.

We construct an instance for the k-Multicolored Independent Matching as follows. First, make a copy of G (with the given coloring of k colors). Then, construct a set $U = \{u_1, u_2, ..., u_k\}$ such that u_i has color $k + i$. Finally, we connect each $u_i \in U$ to each $v_j \in V$ such that $color(v_j) = i$, i.e., we construct a set $E' = \{(u_i, v_j) | u_i \in U, v_j \in V, 1 \leq i \leq k, 1 \leq j \leq n, i = color(v_j)\}$. (Note that each $u_i \in U$ is connected to nodes in V of exactly one color.) Let the resulting graph be $G' = (V \cup U, E \cup E')$, with each vertex in G' colored with one of the $2k$ colors. We claim that G has a colorful independent set of size k if and only if G' has a colorful independent matching of size k. The details are given as follows.

If G has a colorful independent set $V' \subseteq V$ of size k, we select the k vertices in V' and match them up with the k vertices in U to obtain k edges (in E'). The vertices in V' are independent and no two vertices in U share an edge (i.e., vertices in U are also independent); moreover, by the definition of E', the vertices of these k edges contain color pairs $\{(1, k+1), ..., (i, k+i), ..., (k, 2k)\}$. Therefore, among these k edges, no two edges can have their vertices directly connected (by edges in $E \cup E'$). Hence, these k edges form a colorful independent matching for G'.

If G' has a colorful independent matching of size k, then the k edges must be obtained by matching exactly k vertices of V with the k vertices in U. (Otherwise, assume that two vertices v_i and v_j in V form an edge in the optimal colorful matching. Then we cannot have k edges in the matching. This is because at least two vertices in U, of colors $color(v_i)+k$ and $color(v_j)+k$ respectively, cannot match up with vertices in $V \cup U$ by the definition of E'. Hence, the colorful matching would contain at most $k-1$ edges, a contradiction.) By the definition of colorful independent matching, among the k edges, the k corresponding vertices from V cannot share any edge hence form an independent set for G.

As the reduction takes polynomial time, the theorem is proved. □

Motivated by the above negative result, we take a more closer look at Largest Closest Pair Color-Spanning Set (LCPCS). It is basically a multicolored independent set problem on a unit disk graph: given a set Q of n points, each colored with one of the k colors, centered at each point $q_i \in Q$ with $color(q_i)$, we put a disk with radius γ and with color $color(q_i)$, $D_i(\gamma)$, then we have the corresponding intersection graph $G(\gamma)$ of these disks. (There is an edge between two unit disks $D_i(\gamma)$ and $D_j(\gamma)$ centered at q_i and q_j respectively if and only if $d(q_i, q_j) < 2\gamma$.) The LCPCS problem is exactly the multicolored independent set problem on this unit disk graph when γ is maximized to γ^*. More precisely, $G(\gamma^*)$ has a multicolored independent set of size k if and only if the Largest Closest Pair Color-Spanning Set on Q has a solution value $2\gamma^*$ (i.e., the corresponding LCPCS solution $S \subset Q$ has a largest closest pair of value $2\gamma^*$). Of course, to show the W[1]-hardness of LCPCS, it is only necessary to look at its decision version, i.e., whether there is a colorful subset of points $S \subset Q$ such that the closest pair of S has a value at least r, where r is part of the input.

Marx showed that the k-Independent Set problem on unit disk graphs is W[1]-hard [16]. We can use this result and the standard method (similar to Lemma 2) to have the following lemma.

Lemma 3. *k-Multicolored Independent Set on a unit disk graph is W[1]-hard.*

Proof. The proof can be done by a reduction from k-Independent Set on a unit disk graph which is W[1]-hard [16]. Given a unit disk graph G represented by a set $\mathcal{D}$ of n unit disks, for each disk $D \in \mathcal{D}$, we make k copies of colors $\{1, ..., k\}$, with the same center as D. Let the resulting set of disks be $\mathcal{D}_k$. Similar to Lemma 2, it is easy to see that the intersection graph G of $\mathcal{D}$ has an independent set of size k if and only if the intersection graph of $\mathcal{D}_k$ has a multicolored independent set of size k. The reduction takes $O(k|G|) = O(kn^2)$ time. □

We then have the following theorem regarding Largest Closest Pair Color-Spanning Set.

Theorem 3. *Largest Closest Pair Color-Spanning Set is W[1]-hard.*

Proof. As discussed above, we reduce k-Multicolored Independent Set on a unit disk graph to Largest Closest Pair Color-Spanning Set (LCPCS). Let $\mathcal{D}_k$ be the set of unit disks (with radii γ and each is colored in one of the k given colors) for the corresponding unit disk graph $G_k(\gamma)$. The centers of these disks form the point set Q, where a point $q_i \in Q$ inherits the color of the corresponding disk, i.e., each $q_i \in Q$ is colored in one of the k given colors. It is easy to see that $G_k(\gamma)$ has a multicolored independent set if and only if G has a color-spanning set S whose closest pair is at least 2γ. The reduction obviously takes linear time. □

The above result implies that LCPCS does not admit any FPT algorithm unless FPT=W[1]. One temptation is to apply the same idea to reduce the k-clique problem on a unit disk graph to MDCS. But, unfortunately, this does not work. The reason is that the maximum clique on a unit disk graph is polynomially solvable [4].

5. Closing Remarks

Motivated by the open question of Fleischer and Xu, we studied the FPT tractability of some related matching problems under the color-spanning model. We showed in this paper that most of these problems are polynomially solvable (hence FPT), except one version on graphs which can be considered as a generalization of the multicolored independent set problem. And, motivated by this last problem, we made a connection between Largest Closest Pair Color-Spanning Set (LCPCS) and the multicolored independent set problem on unit disk graphs and were able to show that it is W[1]-hard. The original question on the FPT tractability of Minimum Diameter Coloring-Spanning Set (MDCS), is, unfortunately, still open.

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