The Magic Number Three
P versus NP

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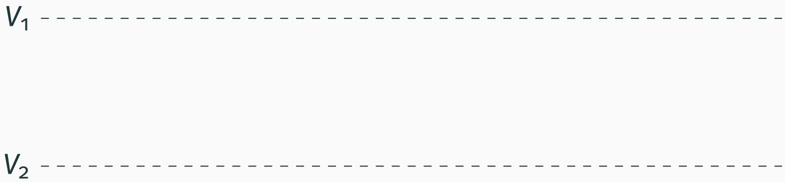
Trier University

One Side Crossing Minimization (OSCM)

Definition (Two-layer drawing)
Let $G = (V_1, V_2, E)$ be a **bipartite** graph.



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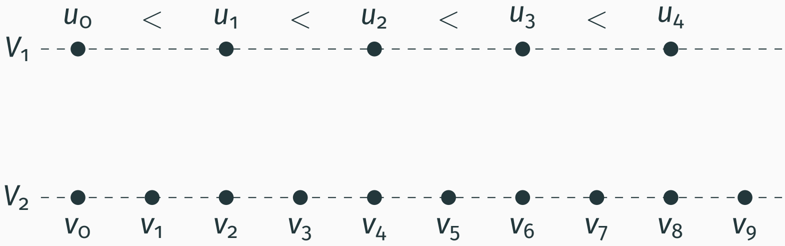

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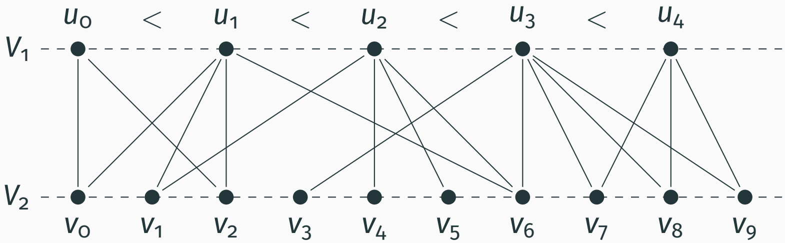

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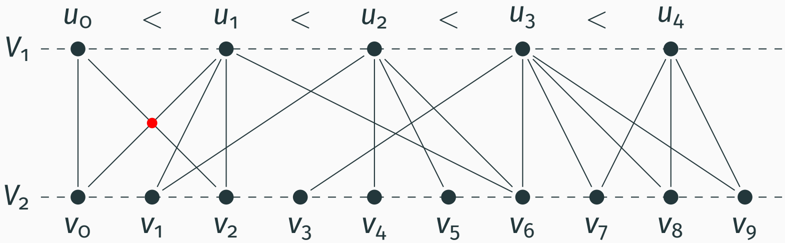

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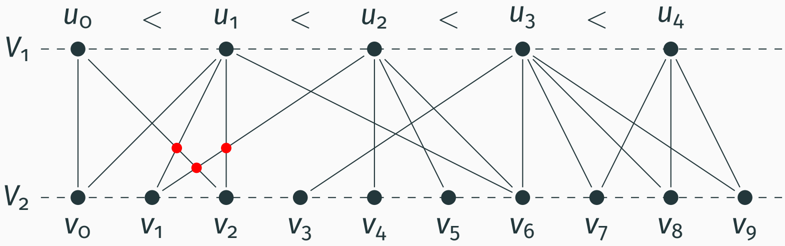

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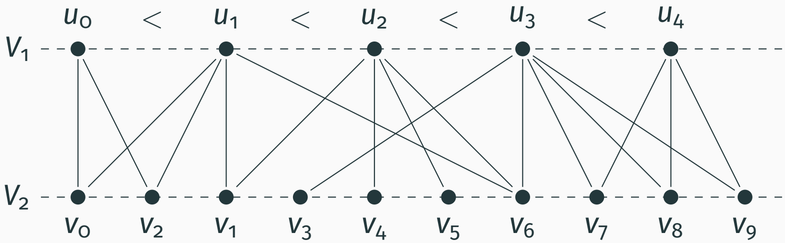

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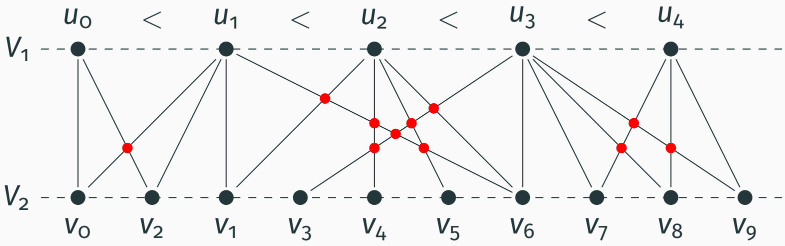

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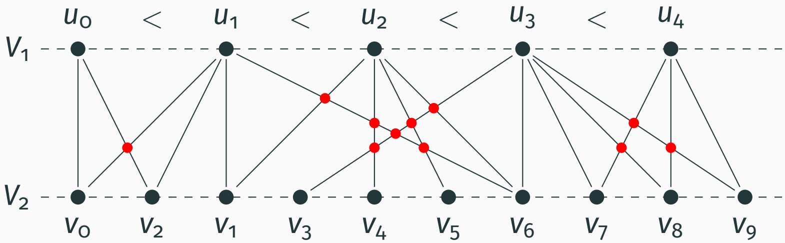

Problem (OSCM)
*Given a **bipartite graph** $G = (V_1, V_2, E)$, a **linear order** τ_1 on V_1 and $k \in \mathbb{N}$. Is there a **linear order** τ_2 on V_2 such that the two-layer drawing specified by (τ_1, τ_2) has at most k edge crossings?*

Grouping by Swapping (GbS)

Definition (Swap)

c a d d a a b c c d d

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c a d d a b ←→ a c c d d

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c a d d a b a c c d d

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Problem (GbS)

*Given a **finite alphabet** Σ, a **string** $w \in \Sigma^*$, and $k \in \mathbb{N}$. Can we transform w in a **blocks string** w' with at most k **swaps**?*

Reduction from GbS to OSCM

GbS
An alphabet $\Sigma = \{a, b, c, d\}$ and $w = caddaabccdd$.

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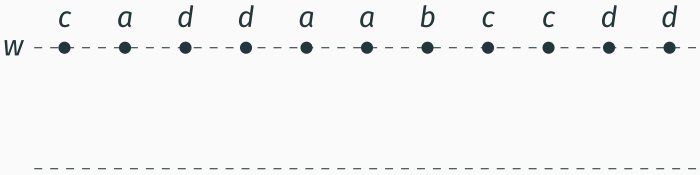

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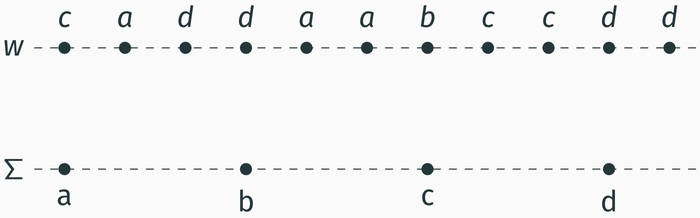

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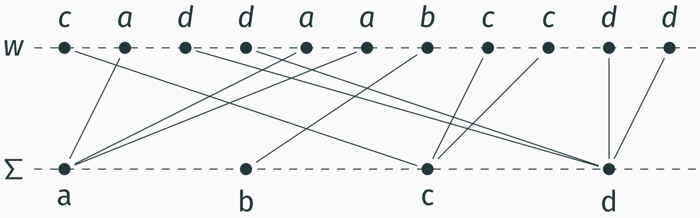

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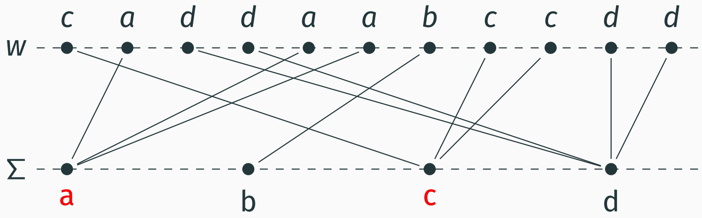

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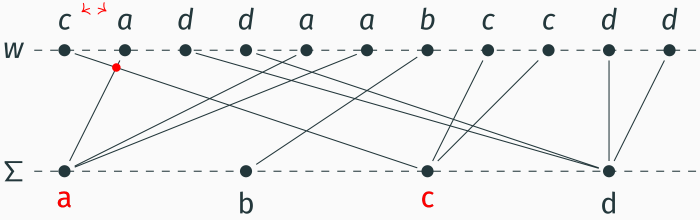

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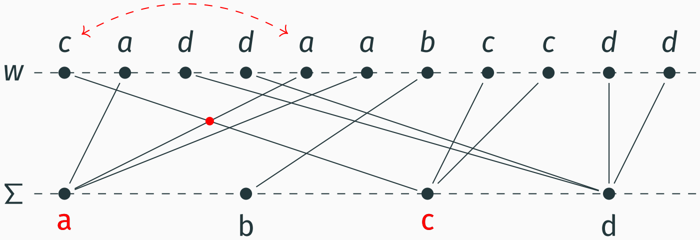

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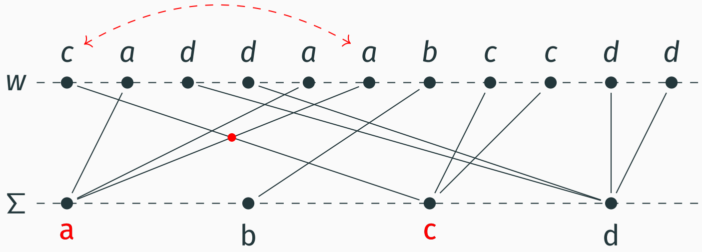

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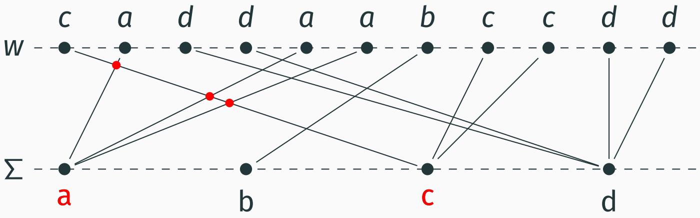

What is known, what is open

OSCM is NP-complete if $G = (V_1, V_2, E)$ is a collection of $K_{1,4}$ with centers of stars in V_2.

OSCM is polynomial-time solvable in graphs of maximum degree 2.

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OPEN: OSCM with a max-3 restriction.

OPEN: GbS if each letter occurs (at most) 3 times.

Recall: Reeb graph presentation!

Kemeny Rank Aggregation (KRA)

Problem (KRA)
Given a ***list of votes*** Π *over a* ***set of candidates*** C *and* $k \in \mathbb{N}$ *Is there a* ***ranking*** τ *on* C *such that the sum of the KT-distances of* τ *from all the votes is at most* k*.*

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\tau: \quad & c_4 < c_7 < c_2 < c_6 < c_1 < c_5 < c_3
\end{aligned}$$

KT-Distance
$(\pi, \tau) = |\{\{i, j\} \mid c_i <_\pi c_j, c_i >_\tau c_j\}|$

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KRA is polynomial-time solvable with 2 voters.

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OPEN: KRA with 3 voters.