Questions

1. **Tight Bound:** Is the inequality

\[ |\ell_1 - \ell_2| \leq \frac{4}{\pi} \cdot (\kappa_1 + \kappa_2) \cdot \mathcal{F}(\gamma_1, \gamma_2) \]

a tight bound for curves in \(\mathbb{R}^n\) for \(n > 3\)?

2. **Simultaneous Scale Space:** What happens if multiple agents are diffusing at the same time?

3. **Understanding \(V_q\):** Does there exist a stability result for \(V_q\)?
Part 1
My Inequality

For curves $\gamma_1$ and $\gamma_2$,

$$|\ell_1 - \ell_2| \leq \frac{4}{\pi} \cdot (\kappa_1 + \kappa_2) \cdot \mathcal{F}(\gamma_1, \gamma_2)$$
Closed Space Curves

A curve is a continuous map $\gamma_i : \mathbb{S}^1 \rightarrow \mathbb{R}^n$. 
Closed Space Curves

A curve is a continuous map $\gamma_i : [0, 1] \to \mathbb{R}^n$, such that $\gamma_i(0) = \gamma_i(1)$. 
Closed Space Curves

A curve is a continuous map $\gamma_i : I \to \mathbb{R}^n$, such that $\gamma_i(0) = \gamma_i(1)$. 

![Diagram of a closed space curve]
Inscribed Polygons
Closed Space Curves

A curve is a continuous map \( \gamma_i : I \to \mathbb{R}^n \), such that \( \gamma_i(0) = \gamma_i(1) \).

\[
\text{mesh}(P) = \max_{0 \leq i < m} (t_{i+1} - t_i)
\]
Arc Length

\[ \ell_i = \ell(\gamma_i) = \int_0^1 \|\gamma_i'(t)\| \, dt \]

\[ \ell(P) = \sum_j \ell(e_j) \]
Arc Length

\[ \ell_i = \ell(\gamma_i) = \int_0^1 \|\gamma'_i(t)\| \, dt \]

\[ \ell(P) = \sum_j \ell(e_j) \]

Lemma

If \( P^k \) is a sequence of polygons inscribed in a smooth closed curve \( \gamma \) such that mesh\( (P^k) \) goes to zero, then

\[ \ell(\gamma) = \lim_{k \to \infty} \ell(P^k). \]
Curvature

Let $x \in I$.
Let $r_x$ denote the radius of the best approximating circle of $\gamma_i(x)$.
Then, the total curvature is:

$$
\kappa_i = \kappa(\gamma_i) = \int_0^1 \frac{1}{r_x} \, dx.
$$
Turning Angle

\begin{figure}
\centering
\begin{tikzpicture}
  \foreach \n in {1,2,3,4,5,6,7,8,9,10}
  \node[circle, draw] (v\n) at (360/10*\n:2cm) {v\n};
  \foreach \a in {9,10}
  \draw (v\a) -- (v\a + 1);\node[scale=0.7] at (v\a) {$v_{\a}$};
\end{tikzpicture}
\end{figure}
Curvature

Let \( x \in I \).

Let \( r_x \) denote the radius of the best approximating circle of \( \gamma_i(x) \).

Then, the total curvature is:

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\]

\[
\kappa(P) = \sum_i \alpha_i.
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Curvature

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Lemma

If $P^k$ is a sequence of polygons inscribed in a smooth closed curve $\gamma$ such that $\text{mesh}(P^k)$ goes to zero, then

$$
\kappa(\gamma) = \lim_{k \to \infty} \kappa(P^k).
$$
The Fréchet Distance

\[ \mathcal{F}(\gamma_1, \gamma_2) = \inf_{\alpha: S^1 \to S^1} \max_{t \in S^1} (\gamma(t) - \gamma(\alpha(t))) \]
Man and Dog
The Fréchet Distance

\[\mathcal{F}(\gamma_1, \gamma_2) = \inf_{\alpha: S^1 \to S^1} \max_{t \in S^1} (\gamma(t) - \gamma(\alpha(t)))\]
The Fréchet Distance

\[ \mathcal{F}(\gamma_1, \gamma_2) = \inf_{\alpha: S^1 \to S^1} \max_{t \in S^1} (\gamma(t) - \gamma(\alpha(t))) \]

Lemma

If \( P^k \) and \( Q^k \) are sequences of polygons inscribed in smooth closed curves \( \gamma_1 \) and \( \gamma_2 \) such that mesh\( (P^k) \) and mesh\( (Q^k) \) go to zero, then

\[ \mathcal{F}(\gamma_1, \gamma_2) = \lim_{k \to \infty} \mathcal{F}(P^k, Q^k). \]
My Inequality

For curves $\gamma_1$ and $\gamma_2$, 

$$|\ell_1 - \ell_2| \leq \frac{4}{\pi} \cdot (\kappa_1 + \kappa_2) \cdot F(\gamma_1, \gamma_2)$$
Two Related Theorems

- [Cha62, F50] For a closed curve contained in a disk of radius $r$ in $\mathbb{R}^n$,

$$\ell_i \leq r \cdot \kappa_i.$$ 

- [CSE07] For two closed curves in $\mathbb{R}^n$,

$$|\ell_1 - \ell_2| \leq \frac{2 \text{vol}(S^{n-1})}{\text{vol}(S^n)} \cdot (\kappa_1 + \kappa_2 - 2\pi) \cdot \mathcal{F}(\gamma_1, \gamma_2).$$
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   a tight bound for curves in \( \mathbb{R}^n \) for \( n > 3 \)?

2. **Simultaneous Scale Space:** What happens if multiple agents are diffusing at the same time?

3. **Understanding \( V_q \):** Does there exist a stability result for \( V_q \)?
Part 2
Scale Space
Scale Space

\[ H(x, 0) \quad H(x, 100) \quad H(x, 1000) \]
The Gaussian

\[ G_1(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t^2}} \]

\[ G_n(x, t) = \frac{1}{(\sqrt{2\pi t})^n} e^{-\frac{|x|^2}{2t^2}} \]

This is the *fundamental solution* to the Heat Equation:

\[ \frac{\partial u}{\partial t}(x, t) - \Delta u(x, t) = 0. \]
Gaussian Convolution

\[ \text{Blur}(y, t, h_0) = \int_{x \in \mathbb{R}^n} G_n(x - y, t) h_0(x) \, dy \]
Homotopy

\[ H : \mathbb{M} \times I \rightarrow \mathbb{R} \] is a continuous function such that

\[ H(x, 0) = f(x) \]
\[ H(x, 1) = g(x) \]
Discrete Homotopy

$H: \text{vert}(K) \times \{0, 1, \ldots, \tau\} \to \mathbb{R}$ is a discrete function such that

$H(x, 0) = f(x)$

$H(x, \tau) = g(x)$
Discrete Homotopy

$H: K \times I_\tau \to \mathbb{R}$ is a discrete function such that

$$H(x, 0) = f(x) \quad \text{and} \quad H(x, \tau) = g(x)$$

and the value at a general point $(x, t) \in M \times I_\tau$ is determined by linear interpolation.
Heat Equation Homotopy

Let $f(x), g(x): \mathbb{R}^2 \to \mathbb{R}$.
Let $h_0(x) = f(x) - g(x)$.
Then:

$$H(y, t) = h_t(y) = \text{Blur}(y, t, h_0) = \int_{x \in \mathbb{R}^n} G_n(x - y, t) h_0(x) \, dy$$

is the solution to the heat equation with initial condition $H(x, 0) = h_0(x)$. 
Let \( f(x), g(x) : \mathbb{R}^2 \to \mathbb{R} \).
Let \( h_0(x) = f(x) - g(x) \).
Then:

\[
H(y, t) = h_t(y) = \text{Blur}(y, t, h_0) = \int_{x \in \mathbb{R}^n} G_n(x - y, t) h_0(x) \, dy
\]

is the solution to the heat equation with initial condition \( H(x, 0) = h_0(x) \).

We define the Heat Equation Homotopy as:

\[
H(x, t) := h_t(x) + g(x).
\]
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a tight bound for curves in \( \mathbb{R}^n \) for \( n > 3 \)?

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Color Images
Color Images

$t = 0$

$t = 100$

$t = 1000$
A **proportion set** for the RGB image is the set of pixels that have the same ratios of colors. For example, the boundaries in the following images depict where $4*\text{blue} = 3*\text{green}$:

- $t = 0$
- $t = 100$
- $t = 1000$
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Persistence Diagrams

A set of points in $\mathbb{R}^2$ that describe the changing homology of the sublevel sets of a function.
Stacking the Persistence Diagrams

We stack the diagrams so that $Dgm_p(h_t)$ is drawn at height $z = t$. 
Then, we match the diagrams using a linear time algorithm [CSEM].
Vineyards

The path of an off-diagonal point is called a vine. A vine is represented by a function $s: I_T \to \mathbb{R}^3$.

The collection of vines is called a vineyard.

Matching of $\text{Dgm}_p(f)$ and $\text{Dgm}_p(g)$ is obtained by looking at the endpoints of the vines.
Another Representation of a Vineyard

[Movie]
Total Movement in a Vineyard

For a vine $s$, we can compute the weighted distance traveled by a point in the persistence diagrams. Then, we sum this distance over all vines in the vineyard $V$.

$$D_s = \int_0^1 \omega_s(t) \cdot \frac{\partial s(t)}{\partial t} \, dt$$

$$V_q(H) = \left( \sum_{s \in V} D_s^q \right)^{1/q}$$
Total Movement in a Vineyard

For a vine \( s \), we can compute the weighted distance traveled by a point in the persistence diagrams. Then, we sum this distance over all vines in the vineyard \( V \).

\[
D_s = \sum_{i \in \{1,2,...,\tau\}} \omega_s(i) \cdot \|s(t_i) - s(t_{i-1})\|_\infty
\]

\[
V_q(H) = \left( \sum_{s \in V} D_s^q \right)^{1/q}
\]
Questions

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3. **Understanding \( V_q \)**: Does there exist a stability result for \( V_q \)?
Let $A = \text{Dgm}(f)$ and $B = \text{Dgm}(g)$. We find a bijection between $A$ and $B$ by minimizing some quantity, such as:

- Bottleneck Distance
- Wasserstein Distance
Bottleneck Matching

The bottleneck cost of a matching is the maximum $L_\infty$ distance between matched points:

$$W_\infty(P) = \max_{(a,b) \in P} \|a - b\|_\infty.$$ 

We seek to minimize the bottleneck distance over all perfect matchings:

$$W_\infty(A, B) = \min_P \{ W_\infty(P) \}.$$
Wasserstein Matching

The Wasserstein cost is measures the cumulative distance as follows:

\[
W_q(P) = \left( \sum_{(a, b) \in P} \|a - b\|_\infty^q \right)^{1/q}.
\]

We seek to minimize the Wasserstein distance over all perfect matchings:

\[
W_q(A, B) = \min_P \{ W_q(P) \}.
\]
Related Stability Results

We say that the matching of persistence diagrams is *stable* if the cost of the matching is bounded by some reasonable function of $||f - g||_\infty$.

- [CSEH] The Bottleneck Distance is stable for monotone Functions $f, g : \mathcal{M} \to \mathbb{R}$.
  \[
  W_\infty(A, B) \leq ||f - g||_\infty
  \]

- [CSEHM10] The Wasserstein Distance is stable for tame Lipschitz Functions with bounded degree $k$ total persistence.
  \[
  W_q(A, B) \leq C^{1/q}||f - g||_\infty^{1-k/q}
  \]
Related Stability Results

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- [CSEH] The Bottleneck Distance is stable for monotone Functions $f, g : \mathbb{M} \to \mathbb{R}$.
  
  $$W_\infty(A, B) \leq \|f - g\|_\infty$$

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  $$W_q(A, B) \leq C^{1/q}\|f - g\|_\infty^{1-k/q}$$

- Is the Vineyard Metric stable too?
  
  $$V_q(f, g) \leq ???$$
Preliminary Findings

Let $A = \text{Dgm}(f)$ and $B = \text{Dgm}(g)$.

$$W_1(A, B) \leq V_1(f, g)$$

$$W_\infty(A, B) \leq V_\infty(f, g)$$
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Questions?
References


