THE TOTAL CURVATURE OF A KNOTTED CURVE

AUTHOR: ISTVÁN FÁRY
TRANSLATED BY: BRITTANY TERESE FASY

The original paper [4] was written in French. Here is a translation.

1. Introduction

1. We are indebted to W. Fenchel [5] for a theorem which is a left closed curve (in ordinary space) has total curvature $\geq 2\pi$. Recently, K. Borsuk [3] gave a new proof of this theorem that applies to curves in $\mathbb{R}^n$. In a note at the end of this paper, Borsuk asked the question wheter the total curvature of a left knotted curve is always $\geq 4\pi$. The primary purpose of this note is to give an affirmative response to the question (see Theorem 5.2).[1] Our demonstration is based on the simple, but interesting, fact that the total curvature of a curve is equal to the average of the its orthogonal projections (see Theorem 3.2).

This remark allows us to also study some other questions, for example the lower bound of the total curvature of curves belonging to the samle class of topological knots, etc. All of these problems can be reduced to questions concerning planar curves; however, as they have a rather combinatorial nature, we do not consider them here. [2]

2. First, we indicate some definitions and we introduce the notations. Consider the *closed curves* in ordinary space. [3] The rectangular coordinates of a variable point of such a curve C is given as a function of the arclength s.[4]

$$C : r(s) = [x(s), y(s), z(s)] \quad (0 \leq s \leq l); \tag{1}$$

Since the curve is closed, one always has $r(0) = r(l)$. We assume that the proposed curves meet the following conditions:

- A1 The curve C has a tangent everwhere (continuously varying), except at most at a finite number of points, corresponding to the values of the parameter $0 \leq d_1 < \cdots < d_k < l$.

[1]Borsuk considers the question regarding regular closed curves, that is to say of closed curves with a well-defined tangent at each point, which varies continuously with respect to the point of contact. We have considered another class of curves (see Introduction, Property 2), but the difference is not essential.

[2]We will come back to these questions mentioned above in another Note, the global curvature of systems of latice curves, etc.

[3]In the space of four dimensions or more, every curve homeomorphic to a circle is isotopic to a circle. So, we consider only ordinary space. We remark here that Theorem 2.1 and Theorem 3.2 are valid in $\mathbb{R}^n$.

[4]This is always possible, since acts only on rectifiable curves.

A2 The second derivitive $r''(s)$ is continuous in each closed interval $d_i \leq s \leq d_{i+1}$.

We use some of the current notations from topology of *gauche curves*.[5] Homeomorphic curves C_0, C_1 are called isotopic if there exists a continuous family of curves C_τ depending on τ, with $(0 \leq \tau \leq l)$, such that C_τ is homeomorphic to C_0 for all $\tau \in [0,1]$. *We say that C is a knot if it is homeomorphic, but not isotopic, to a circle*[6] We use the easily demonstrated fact that if the curve C is a knot, all inscribed polygons to C and having relatively small dimensions are also knots.

Consider three points of the curve C corresponding to the parameters a, b, c and the radius ρ_{abc} of the circle passing through these three points. The curvature at a point s is defined by the limit (supposing it exists)[7]

$$\lim_{d \to 0} \frac{1}{\rho_{abc}} = k(s) \quad (d = |a-s| + |b-s| + |c-s|). \tag{2}$$

We always designate the angle of the vectors a, b by $\Phi(a,b)$, chosen in a way such that

$$0 \leq \Phi(a,b) \leq \pi. \tag{3}$$

One alternate definition of the curvature is the following:

$$k(s) = lim_{|b-a| \to 0} \frac{\Phi(r'(a), r'(b))}{|b-a|} = |r''(s)|. \tag{4}$$

We know that the limits (4) and (4) are equal, if $r''(s)$ is continuous. Finally, we remark that $k(s) \geq 0$ always (even for curves in the plane).

We use the fact that the limits (4) and (4) are uniform on s in each interval $s_1 \leq s \leq s_2$ where $r''(s)$ is continuous.[8]

Finding the total curvature. If $r(s)$ has first and second order derivitives and if the curve is regular and closed, that is to say one has

$$r'(0) = r'(l) \quad [r(0) = r(l)],$$

then the total curvature $K(C)$ is defined by

$$K(C) = \int_o^l k(s)\, ds \tag{5}$$

Furthermore, using 4, one has

$$K(C) = \int |\, d\varphi|, \tag{6}$$

[5]Gauche curves are curves in $\mathbb{R}^n$, but are not planar curves. Specifically, we will talk about curves in $\mathbb{R}^3$ and will use the term curve to mean gauche curve.

[6]A closed curve is without multiple points.

[7]As for the definition of curvature, see [2, 7].

[8]See [6], theorems XXII and XXIII.

designating the infitesimal change in the tangent angle by $d\varphi$. The other part, $|r'(s)| = 1$; $r'(s)$ is consequently on the unit sphere, and $K(C)$ is the arclength of the first spherical image.[9]

This definition extends naturally to more general curves that we conceptualize here, those with properties A1 and A2. It suffices to consider (6) like the Stieltjes integral, that is to say, like the limit

$$\lim_{\max(t_i - t_{i-1}) \to 0} \sum_{i=1}^{n} \Phi(r'(t_i), r'(t_{i-1})) = \int |\, d\varphi| \quad (t_i \neq d_j). \tag{7}$$

Further observe that the total curvature thus defined is not the length of the curve traced on the unit sphere and is composed of images corresponding to continuous segments of $r'(s)$ and of arcs of great circles joining these consecutive acs.

The total curvature of a closed polygon

$$P = A_1, \ldots, A_n (= A_1A_2, \ldots A_{n-1}A_n, A_nA_1) \tag{8}$$

is easily calculated using 7

$$K(P) = \sum_{i=1}^{n} \Phi(\overrightarrow{A_{i-1}A_i}, \overrightarrow{A_iA_{i+1}}) \tag{9}$$

[for closed polyogons we set $A_i = A_j$ for $i \equiv j(\mod n)$].

Finally, note that given the the definition 7 of the total curvature of a curve C, one has the inequality

$$K(C) \geq 2\pi, \tag{10}$$

for the closed planar curves; this inequality is only for convex curves.

2. The total curvature of a curve, considered as the limit of inscribed polygons

Theorem 2.1. *Let P_r be a sequence of polygons inscribed in $C[C : r(s)]$. Suppose that the points of discontinuity of $r'(s)$ are a subset of the vertices of all the polygons P_r, and that the side of maximum length of P_r tends towards zero with $1/r$. Then, the total curvature of P_r thends to the total curvature of C.*

Proof. Consider an arc C' of the curve C, corresponding to the interval $d \leq s \leq d'$, where $r''(s)$ is continuous. Let a, b, c $(d \leq a < b < c \leq d'')$ be three values of s, and define θ and $\theta*$ as follows:

$$\theta = \Phi(r(b) - r(a), r(c) - r(b)), \theta* = \Phi\left(r'(\frac{a+b}{2}), r'(\frac{b+c}{2})\right)$$

[9] This observation leads us to a broader definition of total curvature: if the curve $r'(s)$ is rectifiable and closed, we can define the total curvature of $r(s)$ as the arclength of $r'(s)$ (see [3], pages 251-3). After the Lebesgue theorem, the curvature $x(s)[= |r''(s)|]$ of $r(s)$ exists in this case almost everwhere, and the total curvature is given by formula (5).

We will show that

$$\frac{\theta}{\theta*} \to 1 \tag{11}$$

uniformly on C' as $|c-a| \to 0$.

1° The continuity of $r'(s)$ implies that the anlge θ tends toward zero with $|c-a|$; thus, one has for $|c-a|$ fairly small,

$$\sin\theta \leq \theta \leq (1+\nu)\sin\theta.$$

2° We reported that the limit 4 also takes place in s (on the arc C', where $r''(s)$ is continuous), and we have

$$(1-\nu)k(b)\frac{c-a}{2} \leq \theta* \leq (1+\nu)k(c)\frac{c-a}{2},$$

for $|c-a|$ fairly small.

3° Let $\frac{1}{k*} = \rho_{abc}$ be the radius of the circle through the points $r(a)$, $r(b)$, and $r(c)$ of C'; from (2), one has

$$(1-\nu)k(b) \leq k* \leq (1+\nu)k(b),$$

if $|c-a|$ is sufficently small.

4° Finally, for $|u-v|$ rather small,

$$1-\nu \leq \left|\frac{r(u)-r(v)}{u-v}\right| \leq 1+\nu$$

since s is the arc length, $|r'(s)| = 1$.

Now, suppose we are given ν, and choose $c-a$ small enough for inequalities 1°-4°to hold (the last one for $u = a, v = bandu = b, v = c$) on the arc C'. Dent the center of the circle through the points $r(a)$, $r(b)$, and $r(c)$ by $r*$ and set

$$\theta' = \Phi\left(r(b) - r*, \frac{r(a)+r(b)}{2} - r*\right)$$
$$\theta'' = \Phi\left(r(c) - r*, \frac{r(b)+r(c)}{2} - r*\right)$$

where

$$\theta' + \theta'' = \theta; \quad \theta', \theta'' \geq 0.$$

One has

$$\frac{\sin\theta'}{x*} = \left|\frac{r(b)-r(a)}{2}\right|, \frac{\sin\theta''}{x*} = \left|\frac{r(c)-r(b)}{2}\right|.$$

Using 1°and 2°, we have

$$\frac{\theta}{\theta*} = \frac{\theta'+\theta''}{\theta*} \leq \frac{(1+\nu)(\sin\theta' + \sin\theta'')}{(1-\nu)k(b)\frac{c-a}{2}} = A$$

By applying inequality 3°, we obtain

$$A \leq \frac{(1+\nu)^2 k(b) \left(\left|\frac{r(b)-r(a)}{2}\right| + \left|\frac{r(c)-r(b)}{2}\right|\right)}{(1-\nu)k(b)\left(\frac{b-a}{2}+\frac{c-b}{2}\right)}$$

then, using 4°and the fact that $\frac{p}{q} \leq \frac{p'}{q'}$ leads to

$$\frac{p}{q} \leq \frac{p+p'}{q+q'} \leq \frac{p'}{q'}$$

and we derive the inequality

$$\frac{\theta}{\theta *} \leq \frac{(1+\nu)^3}{1-\nu}.$$

An analagous calculation gives us

$$\frac{\theta}{\theta *} \geq \frac{(1-\nu)^3}{1+\nu},$$

and these last two inequalities give us (11).

Suppose that $r''(s)$ is everywhere continuous. In order to compare the total curvature of the polygon P_r with that of C, we designate by A_{r_i} its vertices and by

$$\theta_{r_i} = \Phi(\overrightarrow{A_{r_{i-1}}A_{r_i}}, \overrightarrow{A_{r_i}A_{r_{i+1}}}) \quad (i = 1, \ldots, n_r),$$

its *exterior* angles whose sum is equal to $K(P_r)$. From (10) we have for r large enough

$$(1-\epsilon)\sum_{i=1}^{n_r} \theta^*_{r_i} \leq \sum_{i=1}^{n_r} \frac{\theta_{r_i}}{\theta^*_{r_i}}\theta^*_{r_i} \leq (1-\epsilon)\sum_{i=1}^{n_r}\theta^*_{r_i}$$

and designate by $\theta^*_{r_i}$ the angle of the tangents of consecutive arcs $A_{r_{i-1}}A_{r_i}$, $A_{r_i}A_{r_{i+1}}$ (with $\epsilon > 0$ is determined in advance). But, according to (7)

$$\sum_{i=1}^{n_r} \theta^*_{r_i},$$

is a sum approximating $K(C)$, which completes the proof in this particular case.

As for the general case, it suffices to subdivide C into partial arcs C_k by the points of discontinuity of $r'(s)$. The ends of C_k are among the ends of P_r, and decompose into partial polygons. One sees that the total curvature of these polygons tends to that of C_k, the angle of the sides separated by the ends of C_k tend toward the angle between the left and right tangents of the points.

Finally, note that if the total curvature of polygons is defined by (9), the total curvature of a general curve C can be defined by the polygons inscribed in C, as the length of a curve is described by the inscribed polygons. Theorem 2.1 shows that this definition is confused with the classic definition for a class of large enough curves [2, p. 17-8]. □

3. The total curvature of a curve, considered as an average of orthogonal projections

Lemma 3.1. *Let a_n, b_n be orthogonal projections of the vectors a and b onto the plan whose normal vector is n. Let*

$$\Phi^*(n; a, b) = \Phi(a_n, b_n).$$

One has

$$\theta = \Phi(a,b) = \frac{1}{4\pi} \iint_{\mathbb{S}} \Phi^*(n; a, b)\, d\omega \tag{12}$$

where $d\omega$ designates the area element of the unit sphere $\mathbb{S}$.

Proof. Let us first show that the integral in (12) only depends only on the angle of the vectors a and b.

For $b = \lambda \cdot a$ with $\lambda > 0$, both members of (12) are zero.

For $b = -\lambda \cdot a$ with $\lambda > 0$,, one has $\Phi^*(n; a, b) = \pi$ (except for $n = \pm\mu a$, which does not change the integral). We have

$$\pi = \frac{1}{4\pi} \iint_{\mathbb{S}} \Phi^*(n; a, -\lambda a)\, d\omega \quad (\lambda > 0)$$

In the general case, where the vectors a and b are linearly independent, let n_0 be the normal to the plane defined by a and b. Let a', b' be other vectors subject to the condition:

$$\Phi(a', b') = \Phi(a, b)$$

We bring $\frac{a}{a}$ to $\frac{a'}{a'}$ and $\frac{b}{b}$ to $\frac{b'}{b'}$ by rotating around the origin (considering fixed points of the vector). In this transformation via rotating the integral (12), one sees that (12) only depends on the angles of the vectors a and b, that is to say,

$$\frac{1}{4\pi} \iint_{\mathbb{S}} \Phi^*(n; a, b)\, d\omega = f(\theta), \quad \theta = \Phi(a, b) \tag{13}$$

We show that the function $f(\theta)$ is a solution of the following equation:

$$f(\theta_1 + \theta_2) = f(\theta_1) + f(\theta_2) \quad (0 \le \theta_1 + \theta_2 \le \pi; \theta_i > 0) \tag{14}$$

Consider three coplanar vectors, a, b, and c, chosen so that

$$\theta_1 = \Phi(a, b), \quad \theta_2 = \Phi(b, c), \quad \theta_1 + \theta_2 = \Phi(a, c).$$

We then have

$$\Phi^*(n; a, c) = \Phi^*(n; a, b) + \Phi^*(n; b, c),$$

and the integration over the unit sphere gives us (14). In addition, we have $0 \le f(\theta) \le \pi$ $(0 \le \theta \le \pi)$.

Given (13) $f(\pi) = \pi$ so $f(\theta) = \theta$ with $0 \le \theta \le \pi$. □ [10]

[10] Theorem 2.1 can be generalized, but we will not deal with the generalization here. We will return to these questions in another Note.

Theorem 3.2. *Let C_n be the orthogoanl projection of the curve C on the plane of the normal vector n. We let $K(C)$, $K(C_n)$ be the total curvatures of C and C_n respectively. If $K(C_n) \leq c$ (independent of n), then one has*

$$K(C) = \frac{1}{4\pi} \iint_{\mathbb{S}} K(C_n)\, d\omega, \tag{15}$$

the integral being extended in all directions, that is, over the whole sphere $\mathbb{S}$.

Proof. This theorem is a consequence of Theorem 2.1 and of Lemma 3.1. For the proof, choose a sequence of polygons P_r inscribed in C, for which we have

$$\lim_{r\to\infty} K(P_r) = K(C). \tag{16}$$

This implies that

$$\lim_{r\to\infty} K(P_{rn}) = K(C_n). \tag{17}$$

where P_{rn} denotes the projection of P_r on a plane with normal n.

Given (17), we have

$$\Im = \frac{1}{4\pi} \iint_{\mathbb{S}} K(P_{rn})\, d\omega = \frac{1}{4\pi} \iint_{\mathbb{S}} \lim_{r\to\infty} K(P_{rn})\, d\omega,$$

and given the Lebesgue theorem:

$$\Im = \lim_{r\to\infty} \frac{1}{4\pi} \iint_{\mathbb{S}} K(P_{rn})\, d\omega = \lim_{r\to\infty} K(P_r) = K(C),$$

This completes the proof.[11] ∎

4. The total curvature of a closed curve

Using Formula (15), we can show:

Theorem 4.1 (de M. Fenchel)**.** *Let C be a closed space curve, and denote by $K(C)$ the total curvature. We then have the inequality $K(C) \geq 2\pi$, where the equality is only valid for convex curves in the plane.*

Proof. Let us first prove that

Proposition 4.2. *If $K(C) = 2\pi$, all the projections of C are of convex curves (that have then the total curvature 2π.*

Suppose the proposition is not correct, and C_{n_0} is a projection of C (on a plane with normal vector n_0), which is not convex. Then there is a neighborhood of n_0 such that C is not convex for each n belonging to this neighborhood (i.e., the limit of a sequence of convex curves is a convex curve). We then have the inequality $K(C_n) > 2\pi$ in this neighborhood of n_0. From (10) and the inequality above, Formula (15) provides the inequality $K(C) > 2\pi$, contradicting the hypothesis. The proposition has now been proven.

[11] As for the assumption $K(C_n) \leq c$ of Theorem 3.2, note that there are curves for which $K(C) < \infty$ and $K(C') = \infty$, where C' is the projection of C.

Now, we prove the theorem of M. Fenchel. The first proposition of the theorem is an immediate consequence of (10) and (15), we need only consider teh case $K(C) = 2\pi$. Using the proposition, we prove that C is a convex plane curve in this case.

Let P, Q, R be any three points of the curve C, and let S a point on the segment PR. Consider the projection C' of the curve C on a plane with normal vector SQ. Designate by $P', Q', R', S'(= S')$ the projections of P, Q, R, S respectively. According to the proposition, C' is a convex curve. The points P', Q', R' are colinear, that is to say C' contains the segment $P'Q'$. The arc P, Q, R o C is contained in a plane. As is true for every set of three points on the curve C, the latter is also contained in a plane. Now, from (10), C is a convex curve, which proves the theorem. □

5. The Inequality $L(C) \geq 4\pi$ for knots

In the first part of this chapter we consider projections of a closed polygone without multiple points au point de vue combinatoire, et nous a closed polygon without multiple points (in a combinatorial point of view), and we have given the necesary criteria to recognize if the polygon is a knot.

In what follows, let P be a closed polygon without multiple points. A projection P' of P is said to be regular, if the projection brings together more than two sides of P.

Proposition 5.1. *Let P be a knotted polygon, and P' one of the regular projections on the plane Π. There exists a point O of the plane Π, as all the rays starting at O and going to Π cut P' in $k \geq 2$ different points (or a multiple point).*

Proof. Since P' is a regular projection, its double points belonging to two of its sides are different vertices and are located on the brder of three or four regions. The polygon P' decomposes the plane into the regions:

a. The regions are divided into two classes $\mathcal{U}$ and $\mathcal{V}$ such that for each region $U \in \mathcal{U}$ and $V \in \mathcal{V}$, U and V have no sides in common [1].

   Let U_0 be the region containing the point at infinity, and let $U_0 \in \mathcal{U}$. Denote the regions neighboring U_0 by $V_1, \ldots, V_{i_1}$ with $V_i \in \mathcal{V}$ for $1 \leq i \leq i_1$. Let $U_1, \ldots U_{i_1}$ be the regions which are adjacent to one or more of V_i with $1 \leq i \leq i_1$, etc. Show that each domain belongs to only one of the classes $\mathcal{U}$ or $\mathcal{V}$; otherwise, we have a broken closed line L which has an odd number of poins in common with P'. Deforming L, the number of intersection points with P' varies, but remains odd. When L is reduced to a point, however, this number is zero. This contradiction proves proposition a.

b. If p' is a regular projection of a knot, there is a region disjoint from the region containing the point at invinity (we have $U_1 \neq U_0$).

   Suppose instead that all the different regions of U_0 have a side in commen with it, that is to say *adjacent*. Let us show that in this case we can deform P into a polygon P_1, with a projection P_1' on the plane Π, which contains the point at infinity.

Region V_1 of P' is in the gerion V_{k_t} with a double point of P' matched to V_1; the region $V_{k_{r+1}}$ is defined in the same way after V_{k_r} (the construction is not unique of course). If $V_{k_m} = V_{k_n}$, one can construct a broken, closed line, in which the inside after a contains a region separate from the one containing the point at infinity. This case is excluded, one reaches a region V_k, for which $V_{k_{st}}$ does not exist, that is to say, such that its border is a noose, having a double point of P'. By a suitable deformation, one can remove the noose of projections of P'.

We can thus deform P into a polygon $\bar{P}$, for which the projection $\bar{P}'$ is only a finite region, $\bar{P}$ is the isotop of a circle. b is then completely shown.

Choose the point O in the interior of $U_1 (\neq U_0)$. We see immediatly that O has the property: all of the rays begining at O intersect P' in $k \geq 2$ different points (or in one multiple point). The proof of the proposition is now complete. ◻

Theorem 5.2. *Each knot C has total curvature $\geq 4\pi$.*

Proof. Let P' be a regular projection. I say that

$$K(P') \geq 4\pi. \tag{18}$$

Let $P' = A_1, \dots, A_n$. Choose the point O having the property of the proposition, and so the points OA_sA_t with $s \neq t$, $1 \leq s, t \leq n$ are not aligned. Let

$$\begin{aligned} \gamma_k &= \Phi(\overrightarrow{OA_k}, \overrightarrow{OA_{k+1}}), \\ \theta_k &= \Phi(\overrightarrow{A_{k-1}A_k}, \overrightarrow{A_kA_{k+1}}), \quad [A_i = A_j, if i \cong j (\mod n)], \\ x_k &= \Phi(\overrightarrow{A_kO}, \overrightarrow{A_kA_{k+1}}), \end{aligned}$$

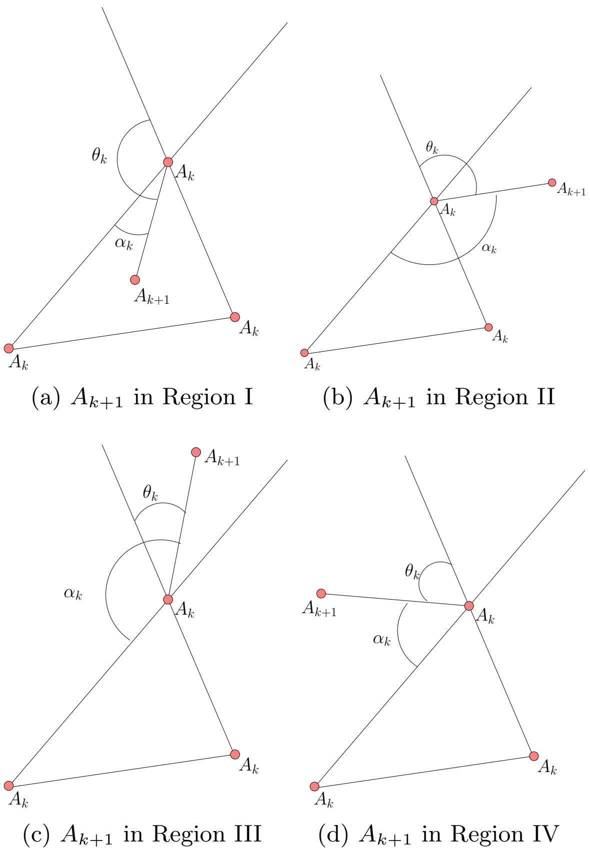
Given the definition of O, one has the inequality:

$$\sum_{k=1}^{n} \gamma_k \geq 4\pi \tag{19}$$

We will prove

$$\theta_k \geq x_{k-1} - \gamma_{k-1} - x_k \quad (k = 1, \dots, n). \tag{20}$$

The lines OA_k and $A_{k-1}A_k$ decompose the plane into four areas (these lines are different from the choice of O). Denote by I the domain containing the triangle $OA_{k-1}A_k$ (which determines the positive direction of the plane), and let II, III, and IV be the other areas taken in the positive direction. For the value of θ_k we obtain (see Figure 1)



(a) A_{k+1} in Region I (b) A_{k+1} in Region II

(c) A_{k+1} in Region III (d) A_{k+1} in Region IV

Figure 1: Geometric illustration of θ_k and α_k for various placements of A_{k+1}.

$$\begin{aligned}
\theta_k &= \alpha_{k-1} + \gamma_{k-1} + x_k && \text{if } A_{k+1} \in I, \\
\theta_k &= 2\pi - (\alpha_{k-1} + \gamma_{k-1} + x_k) && \text{if } A_{k+1} \in II, \\
\theta_k &= -\alpha_{k-1} - \gamma_{k-1} + \alpha_k && \text{if } A_{k+1} \in III, \\
\theta_k &= \alpha_{k-1} + \gamma_{k-1} - \alpha_k && \text{if } A_{k+1} \in IV,
\end{aligned}$$

Innequality (20) is then immediate. Taking the sum, we get

$$\sum_{k=1}^{n} \theta_k \geq \sum_{k=1}^{n} \gamma_k$$

and from (19), we get the inequality (18).

Now, let C be a given knot. If $\epsilon > 0$, we can find a polygon P inscribed in C and a knot, as

$$K(C) + \epsilon \geq K(P).$$

For all regular projections P_n of P, we have $K(P_n) \geq 4\pi$; the singular directions are located in planes and hyperboloids finite in number, and have a measure zero on the surface of the unit sphere. Integration over the surface of the sphere of both sides of (18) gives

$$K(X) + \epsilon \geq 4\pi,$$

and since ϵ is an arbitrary number greater than zero, we obtain Theorem 5.2.

Finally note that we can build knots (belonging to the most simple topological class) with the total curvature $\leq 4\pi + \epsilon$ with $\epsilon > 0$ arbitrary, that is tos say the inequaltiy of Theorem 5.2 is the best.[12]

Added to the proofreading (31 May 1949). – I just received a letter from Mr. Borsuk, who informed me that Theorem 5.2 was proved independently by H. Hopf. He uses the theorem of Mlle. Pannwitz, who ensures that for every knot, one can find the right cut, cutting it at least in four points.

□

References

[1] Bankwitz, A. Ueber die Torsionszahlen der alternierenden Knoten. *Math. Ann.* (1930), 145–6.

[2] Blaschke. *Vorlesungen über Differentialgeometrie.* 1930.

[3] Borsuk, K. Sur la courbure totale des courbes fermées. *Ann. Soc. Pol. Math.* (1948), 251–265.

[4] Fary, I. Sur certaines inégalités géométriques. *Acta Sci. Math. 12* (1950), 117–124.

[5] Fenchel, W. Über Krümmung und Windung geschlossener Raumkurven. *Math. Ann.* (1929), 238–252.

[6] Pauo. Les methodes directes en geometrie differentielle. 124–5.

[7] Poussin, D. L. V. *Cours d'Analyse infinitésimale*, 8 ed. 1938.

[12] Manuscript received 15 February 1949.