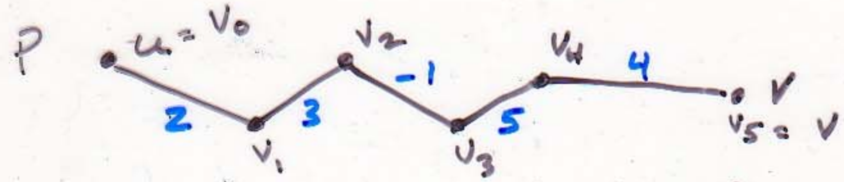
CS 223 Lec 21

Shortest Path Problems

$G = (V, E)$ (un)directed

$w : E \to \mathbb{R}$



$(u \overset{P}{\leadsto} v)$

path from u to v

$P = \langle v_0, v_1, \ldots, v_5 \rangle$

$$w(P) = \sum_{i=1}^{n} w(v_{i-1}, v_i)$$

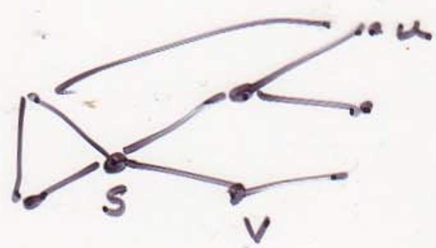
weight of example = 13

Shortest Paths

$\delta(u, v)$ = shortest path distance from u to v

$$= \begin{cases} \min \{ w(P) : u \underset{P}{\leadsto} v \} \\ \infty \quad \text{(if no path exists)} \end{cases}$$

<u>Problem</u> Given s (source vertex) find the shortest path from s to all other vertices. (single-source shortest path)



<u>Variation 1</u>

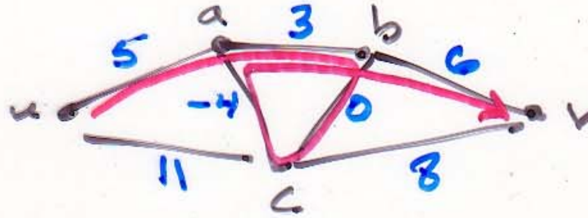
single-pair shortest path problem

single source, destination

<u>Variation 2</u>

all pairs shortest path.

Negative Weights



$$\delta(u,v) = -\infty$$

(no limit on how low shortest path wt can go)

$\Rightarrow$ Dijkstra's alg can't handle this

$\Rightarrow$ Bellman-Ford can handle this.

Information

$d[v]$ = "upper bound" estimate on $\delta(s,v)$

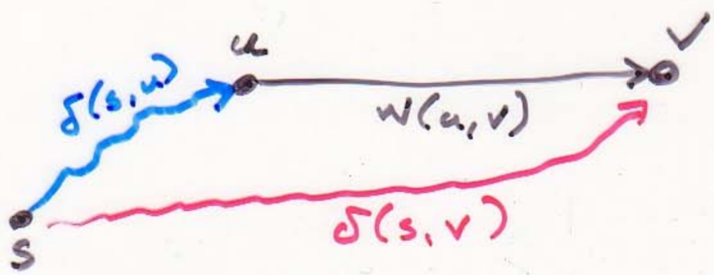
$\pi[v]$ = predecessor of v on a shortest path from s to v

Properties

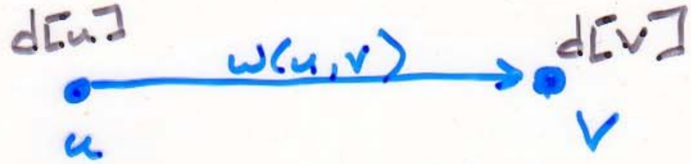
1) $d[v] \geq \delta(s,v)$

2) Triangle Inequality

$$\delta(s,v) \leq \delta(s,u) + w(u,v)$$



Edge Relaxation



RELAX (u, v)

  if $d[v] > d[u] + w(u,v)$

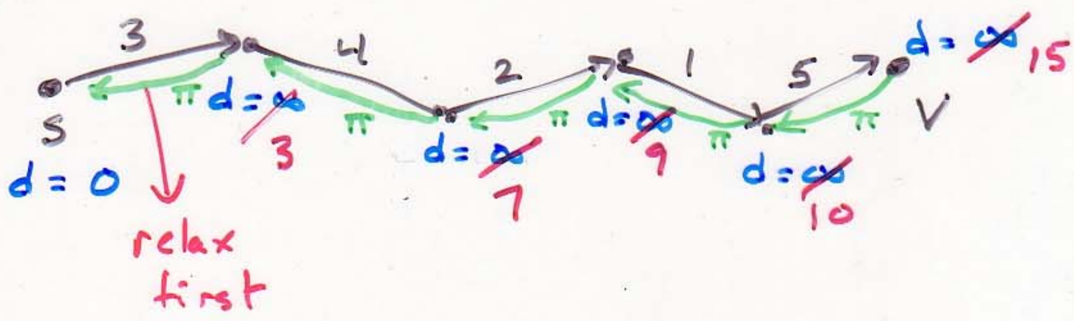
  then {

    $d[v] = d[u] + w(u,v)$

    $\pi[v] = u$

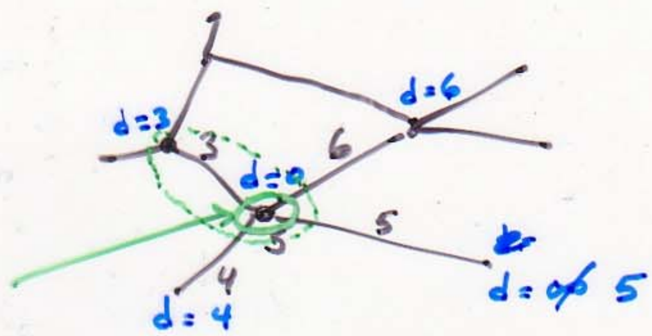
  }

suppose this is the shortest path



<u>Note</u>: if we relax edges along a shortest path from s to v in order then $d[v] = \delta(s,v)$ and following π pointers from v constructs the path.

Dijkstra's Algorithm



S
=
set of vertices for which
a shortest path is
known

DIJKSTA (G, w, s)

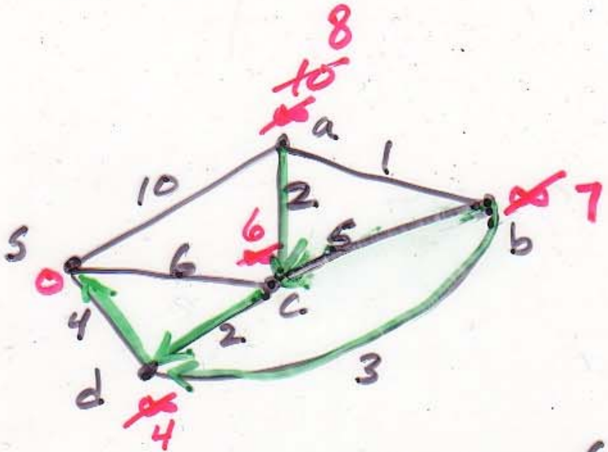
- initialize d, π values
  $(d(s) = 0,\ d(v) = \infty$
  $\quad v \neq s$
  $\pi(v) = \text{nil}\ )$

- $S = \emptyset$
- $Q = V$ (min priority queue)
- while $Q \neq \emptyset$ {

  u = EXTRACT-MIN (Q)
  $S \leftarrow S \cup \{u\}$
  for each $v \in Adj(u)$
  relax (u, v)

}

Example



d red

$Q = \{ \overset{0}{\cancel{s}},\ \overset{10}{\cancel{\infty}}\!a,\ \overset{\infty}{b},\ \overset{6}{\cancel{\infty}}\!c,\ \overset{4}{\cancel{\infty}}\!d \}$

$S = \{s\}$

$Q = \{ \overset{8\,\cancel{10}}{a},\ \overset{9}{\cancel{b}},\ \cancel{c} \}$ $\quad S = \{s, d, c\}$

$S = \{s, d, c, b, a\}$

$Q = \{\cancel{a}\}$