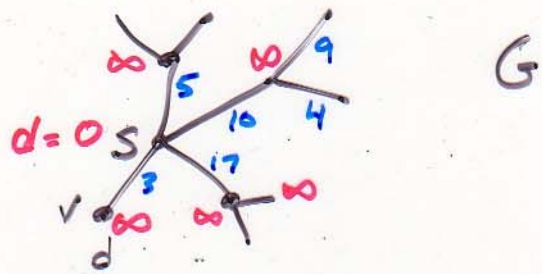
CS 223 Lec 22

Correctness of Dijkstra's algorithm



d

upper bound on $\delta(s, v)$

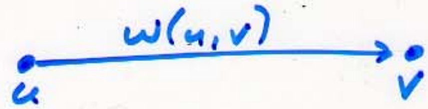
true shortest path distance from s to v

RELAX(u, v, w)

if $d[u] + w(u,v) < d[v]$

then

$$d[v] = d[u] + w(u,v)$$

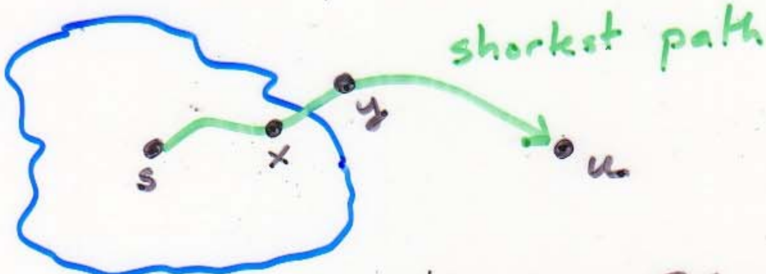


Pf **idea** when a vertex v is added to S

$$d[v] = \delta(s, v)$$

assume this is not true

let u be a vertex about to be added to S where $d[u] \neq \delta(s,u)$



S

$$d[x] = \delta(s,x)$$

since $x \in S$, (x,y) was relaxed.

So

$$\begin{aligned} d[y] &\le d[x] + w(x,y) \\ &= \delta(s,x) + w(x,y) \\ &= \delta(s,y) \end{aligned}$$

$$\Rightarrow d[y] = \delta(s,y)$$

Observe

$$\begin{array}{ccc} \delta(s,y) & \le & \delta(s,u) \\ \parallel & & \wedge\!\vert \\ d[y] & & d[u] \end{array}$$

but u picked next, $d[u] \le d[y]$

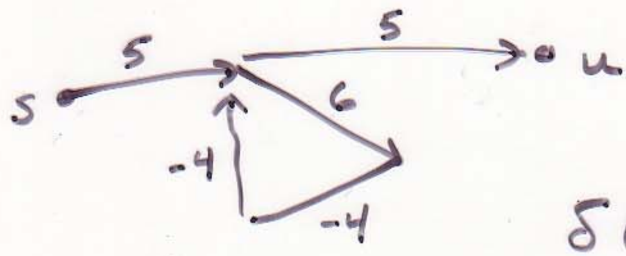
So it must be that

$$d[y] = \delta(s,y) = \delta(s,u) = d[u]$$

$$\Rightarrow d[u] = \delta(s,u)$$

this means u doesn't contradict our assumption, so Dijkstra's alg. is correct.

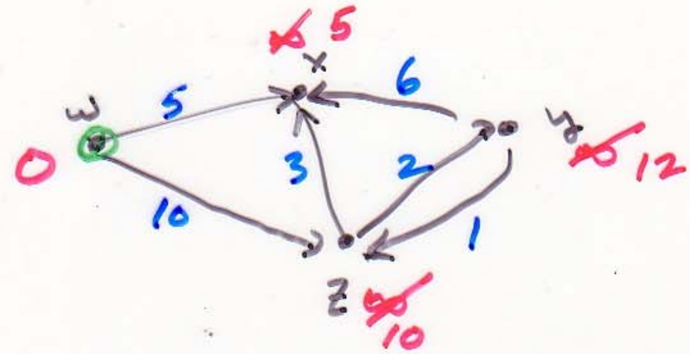
<u>What if we have negative wt. cycles?</u>



$\delta(s,u) = -\infty$

<u>Bellman</u> - <u>Ford</u> <u>Algorithm</u>

Suppose G does not contain a neg. wt. cycle.

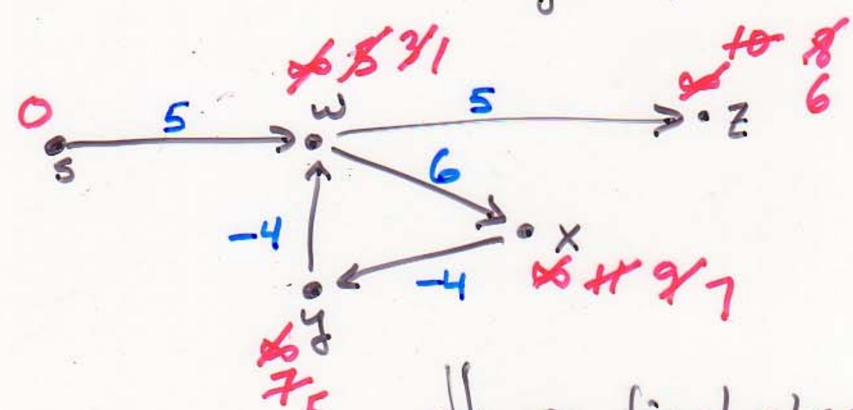


$d[w] = 0 \qquad d[v] = \infty$

$v \neq w$

Suppose G does have a neg. wt. cycle?

iterations ||||



$\Downarrow$ on final check this edge is detected

$\Downarrow$

returns FALSE

(G contains a neg. wt cycle)

Application of BellMAN-Ford

arbitrage = currency trading

We have currencies

$c_1, c_2, \ldots c_n$

Exchange Rates

1 unit of c_i buys $R[i,j]$ units of c_j

example

\$ 1 USD buys 46.4 Indian rupees

1 rupee buys 2.5 Japanese yen

1 yen buys 0.0091 USD

$1 \times 46.4 \times 2.5 \times 0.0091$

$= 1.0556$ USD

Question: is there

a sequence of currencies

$\langle c_{i_1}, c_{i_2}, \ldots, c_{i_k} \rangle$

so that

$$\log\left(R[i_1,i_2]\cdot R[i_2,i_3]\cdot \ldots R[i_{k-1},i_k]\cdot R[i_k,i_1]\right)$$

$$\underset{\log}{>}(1)\,?$$

$$\log R[i_1,i_2] + \log R[i_2,i_3] + \ldots + \log R[i_k,i_1]$$

$$> 0\ ?$$

<u>idea</u> is to create a <u>graph</u>

on the currencies

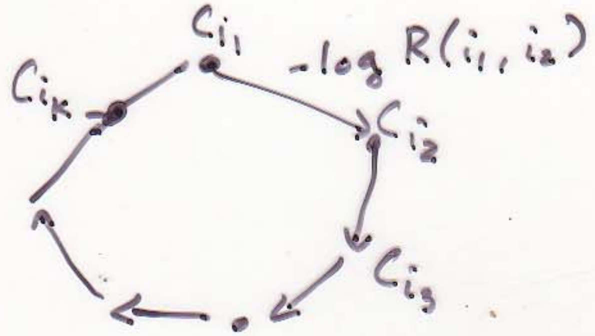
where

$$w(c_i, c_j) = -\log R(i,j)$$

if B-F detects a neg. wt. cycle <u>then</u>

there is a cycle of currencies $\langle c_{i_1}, c_{i_2}, \dots c_{i_k} \rangle$

such that



$$-\log R(i_1, i_2) + -\log R(i_2, i_3) + \dots + -\log R(i_k, i_1) < 0$$

$$\Rightarrow \log R(i_1, i_2) + \dots + \log R(i_k, i_1) > 0$$

$$\Rightarrow R(i_1, i_2) \cdot \dots R(i_k, i_1) > 1$$