CS 222 Lec 13

Relations + Functions cont.

Binary Relations

$$R \subseteq A \times B$$

$$R = \{(x, y) \mid x, y \in \mathbb{Z} \quad x - y \text{ is divisible by } 4\}$$

$(4, 0) \in R$

$(5, 1)$

$(8, 0)$

$R \subseteq \mathbb{Z} \times \mathbb{Z}$

"R is a relation on $\mathbb{Z}$"

$A = \{\text{students}\}$

$B = \{\text{classes}\}$

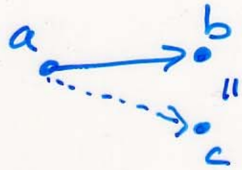
$$R = \{(x, y) \mid x \in A,\ y \in B \quad \text{student } x \text{ is taking class } y\}$$

$$R \subseteq A \times B$$

Functions are binary relations

$$F = \{ (a,b) \}$$

$$F \subseteq A \times B \iff f : A \to B$$



$$(a,b) \in F \text{ and } (a,c) \in F$$

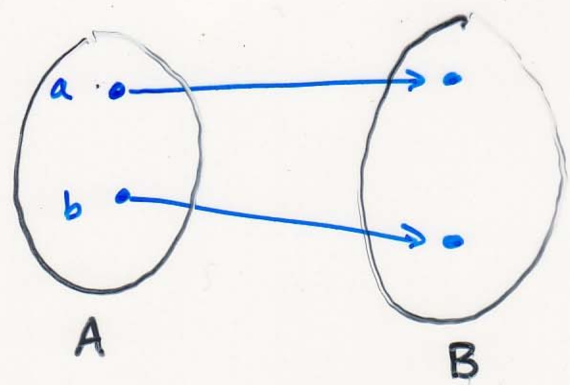
$$\Rightarrow b = c$$

Properties of Functions

Defn A function $f : A \to B$

is one-to-one if

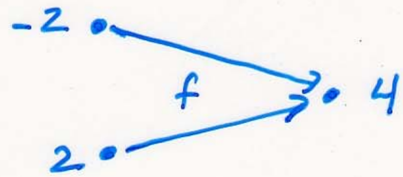
$$a \neq b \Rightarrow f(a) \neq f(b)$$



e.g. $f : \mathbb{Z} \to \mathbb{Z}$

$$f(x) = x + 1$$

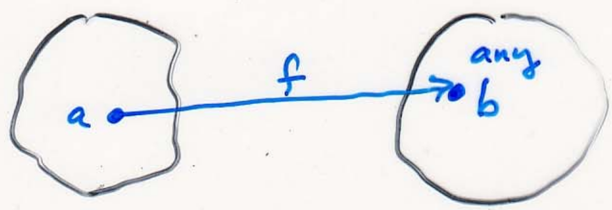
$f(x) = x^2$ is not $\underline{1-1}$:



<u>Defn</u> A function $f: A \to B$

is <u>onto</u> if

$\forall b \in B \ \exists a \in A, \ f(a) = b$



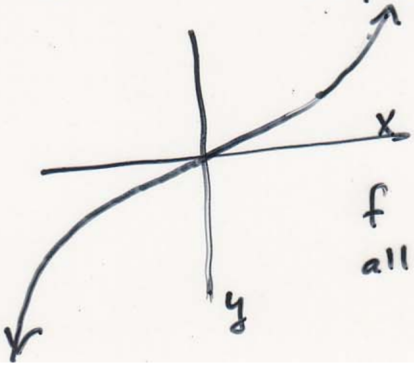
$$f(A) = \{ f(a) \mid a \in A \}$$

if f is onto: $f(A) = B$

e.g. $f(x) = x$

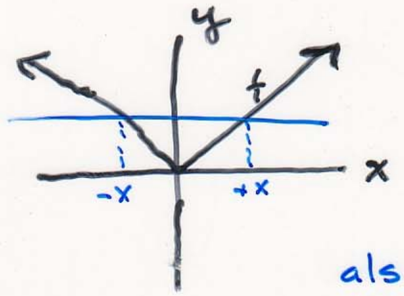
$f(x) = x^3$ if $A, B = \mathbb{R}$

also 1-1.



f is onto, since all y values are 'hit'

$f(x) = |x|$ is not onto:



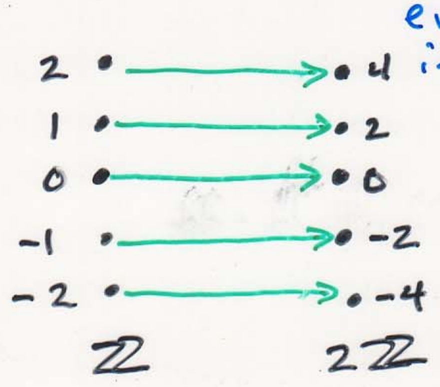
also not 1-1.

<u>Defn</u> A function $f: A \to B$ that is 1-1 and onto is called <u>bijection</u> (or <u>one-to-one</u> <u>correspondence</u>)

<u>e.g.</u> $f: \mathbb{R} \to \mathbb{R}$ $f(x) = x^3$ is a bijection.

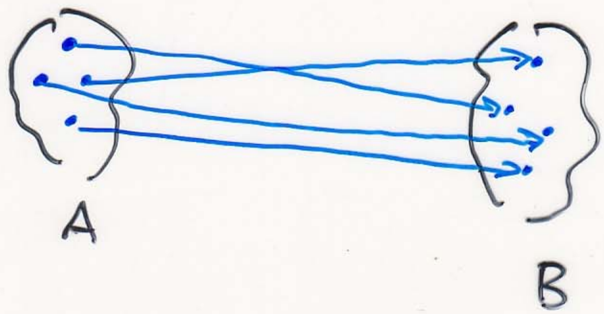
$f: \mathbb{Z} \to \mathbb{Z}$ $f(x) = x+1$

$f: \mathbb{Z} \to \underbrace{2\mathbb{Z}}_{\text{even integers}}$ $f(x) = 2x$



Claim f is invertible

$\Longleftrightarrow$ f is a bijection.



Theorem $f : A \to B$ $|A| = m$

$|B| = n$

1) f is 1-1 $\Rightarrow$ $m \leq n$

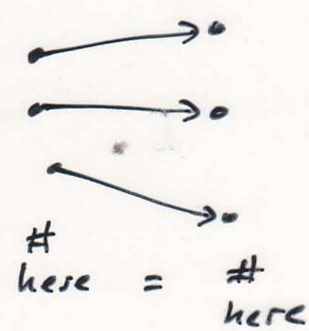
2) f is onto $\Rightarrow$ $n \leq m$

3) f is a bijection $\Rightarrow$ $m = n$

Proof

1) f is 1-1.

Claim $|f(A)| = |A|$



$$f(A) \subseteq B$$

so $$\underset{\substack{\| \\ |A|}}{|f(A)|} \le |B|$$

$$\Rightarrow m \le n$$

2) <u>f is onto</u>

<u>Claim</u> $|f(A)| \le |A|$

f is onto

$$\Rightarrow f(A) = B$$

$$|f(A)| = |B|$$

so $$|B| \le |A|$$

$$\Rightarrow n \le m$$

3) <u>f is a bijection</u> $\Rightarrow$ f is 1-1 and onto

$$\Rightarrow m \le n \text{ and } n \le m$$

$$\Rightarrow m = n$$

Corollary $f: A \to B$, $m > n$
$(|A| = m, |B| = n)$

Then f is not 1-1.

Pigeonhole Principle