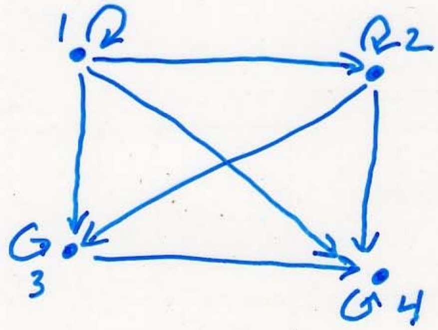
# CS 222 Lec 14

## Properties of Relations

$R_1$ on $\{1,2,3,4\}$

given by $(x,y) \in R_1 \iff x \le y$

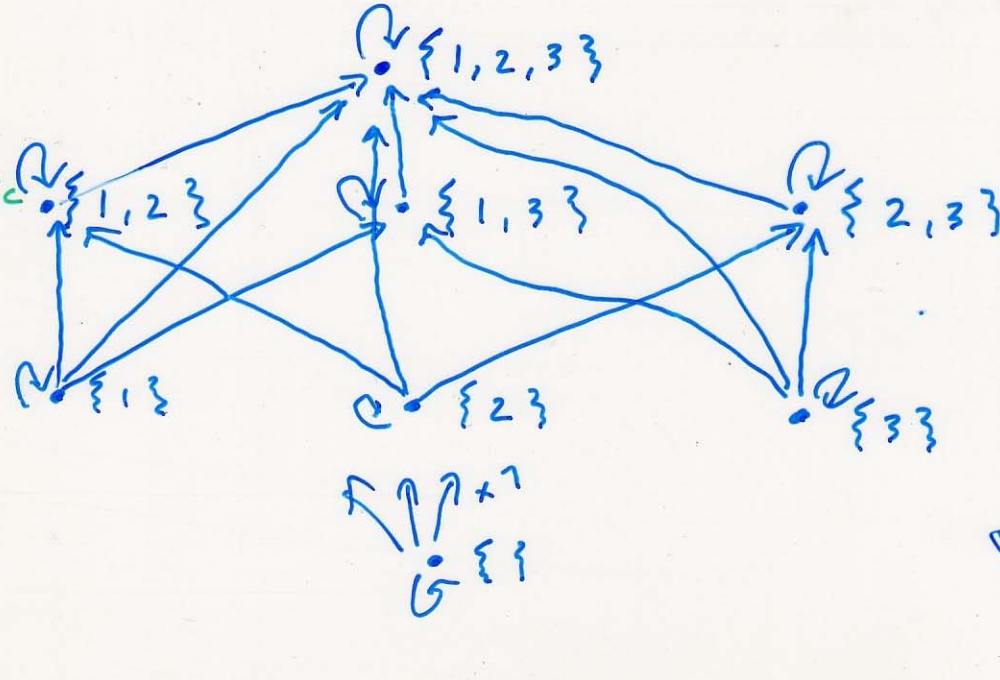



reflexive
antisymmetric
transitive

$R_2$ on $\mathcal{P}(\{1,2,3\})$

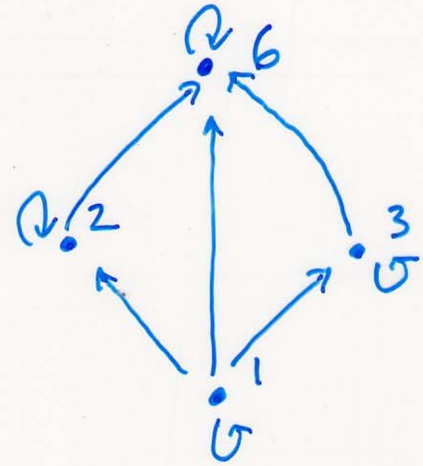
$(x,y) \in R_2 \iff x \subseteq y$

reflexive
antisymmetric
transitive




$R_3$ on $\{1,2,3,6\}$

$(x,y) \in R_3 \iff x \mid y$




reflexive

antisymmetric

transitive

## Properties

Let $R$ be a relation (binary) on a set $A$.

We say $R$ is :

1) <u>reflexive</u> if $\forall a \in A$ $(a,a) \in R$.

2) <u>symmetric</u> if

if $a \neq b$: $(a,b) \in R \Rightarrow (b,a) \in R$

3) <u>antisymmetric</u> if

if $a \neq b$: $(a,b) \in R \Rightarrow (b,a) \notin R$

4) <u>transitive</u> if (all short-cuts exist)

$\forall a, b, c \in A$

$(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow (a,c) \in R$

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend left, dashed, "\text{(short-cut)}"] & b \arrow[r] & c \end{tikzcd}$$

**Defn** A relation $R$ that is reflexive, antisymmetric, + transitive is called a <u>partial</u> <u>order</u>.

**Note**: $R$ can be drawn so that all arrows (not self-loops) point upwards.

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**Defn** A relation $R$ is a <u>strict</u> <u>order</u> if it is not reflexive, and antisymmetric and transitive.

e.g. $A = \mathbb{Z}$, $R = \{(x,y) \mid x < y\}$

$$\cdots < -2 < -1 < 0 < 1 < 2 < \cdots$$

$$\cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to \cdots$$

## Partitions

Let $A$ be a set.

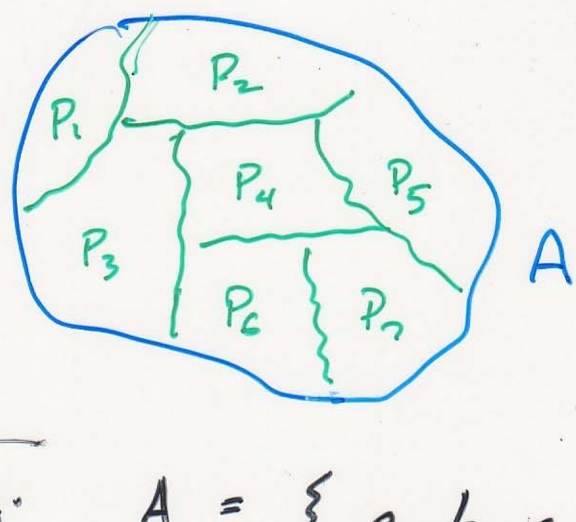
We say that $P = \{P_1, P_2, \ldots\}$ is a partition of $A$ if:

sets

1) $\forall i > 0, \; P_i \neq \emptyset$

2) $P_i \neq P_j \Rightarrow P_i \cap P_j = \emptyset$ (disjoint)

3) $A = \bigcup_{i>0} P_i$

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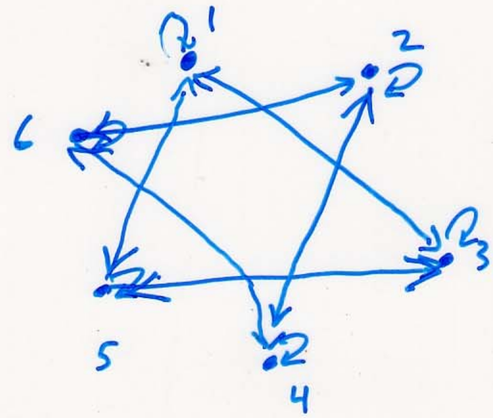
e.g. $A = \{a, b, c, d, e\}$

$P = \{\{a, b\}, \{c\}, \{d, e\}\}$

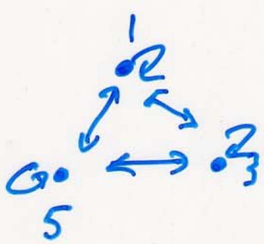
Example

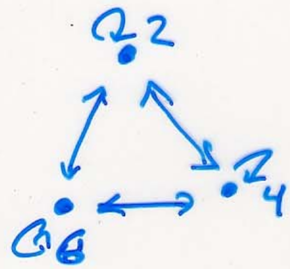
$$A = \{1, 2, 3, 4, 5, 6\}$$

$$R = \{ (a,b) \mid a-b \text{ is even} \}$$




redraw




reflexive

symmetric

transitive.

e.g. $A = \mathbb{Z}$

$$P_1 = \{ \text{even numbers}, \text{odd numbers} \}$$

$$P_2 = \{ \mathbb{Z}^{\geq 0}, \mathbb{Z}^{<0} \}$$

$$P_3 = \{ \ldots, \{-2\}, \{-1\}, \{0\}, \{1\}, \{2\}, \ldots \}$$

$P_2$

$P_3$

$$\ldots, \boxed{\boxed{-3}, \boxed{-2}, \boxed{-1}}, \boxed{\boxed{0}, \boxed{1}, \boxed{2}, \boxed{3}, \ldots}$$

<u>Defn</u> <u>E</u>quivalence <u>Rel</u>ations

Let $R$ be a relation on $A$.

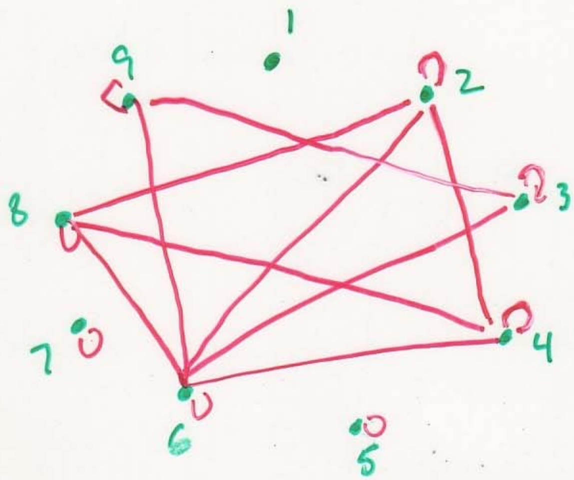
We say $R$ is an <u>equivalence relation</u> if $\exists$ a partition $P$ of $A$ s.t.

$(x, y) \in R \iff x$ and $y$ belong to the class in $P$.

Examples $A = \{1, \dots 9\}$

$R = \{(a,b) \mid \gcd(a,b) > 1\}$

$P = \{$ <u>not</u> an equivalence relation $\}$

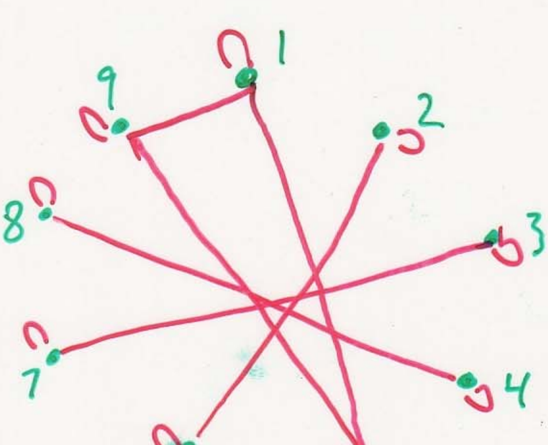



$\{3,6,9\} \cap \{2,4,6,8\} = \{6\}$

$\overset{\times}{=} \emptyset$

$A = \{1, \dots, 9\}$

$R = \{(a,b) \mid a-b$ is divisible by 4 $\}$




$P = \{ \{1,5,9\},$
$\{2,6\},$
$\{3,7\},$
$\{4,8\} \}$

R <u>is</u> an equivalence relation

Thm A relation $R$ on $A$ is an equivalence relation $\Leftrightarrow$ $R$ is reflexive, symmetric and transitive.

Defn The classes in the partition are called equivalence classes.