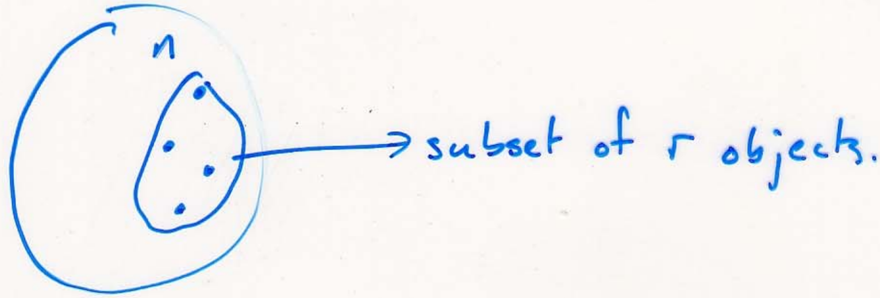
CS 222 Lec 17

Combinatorics cont.

Combinations

n objects

# of ways we can <u>select</u> r of them



$$C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

read "n choose r"

$$\binom{10}{3} = \frac{10!}{3!\,7!} = \frac{10\cdot \cancel{9}^{3}\cdot \cancel{8}^{4}}{\cancel{3}_{1}\cdot \cancel{2}_{1}\cdot 1} = 120$$

order matters?

repeats allowed?	yes	no
yes	ordered list n^r	bag/multiset
no	permutation $P(n,r)$	combination $C(n,r)$

$$P(n,r) = \frac{n!}{(n-r)!}$$

Combinations Examples

52 cards $\longrightarrow$ 5 card hand

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

# of 5-card hands containing one Ace?

pick hand:

1. choosing an Ace
2. choosing 4 remaining cards.

$$\# \text{ways} = \binom{4}{1} \cdot \binom{48}{4}$$

# of 5-card hands with 2 Aces?

$$\# \text{ ways} = \binom{4}{2}\binom{48}{4}$$

# of 5-cards hands with ≥ 1 a pair?

$\binom{13}{1}$ 1. choose which ~~card~~ value is duplicated

$\cdot$

$\binom{4}{2}$ 2. choose the two cards that have that value

$\cdot$

$\binom{48}{3}$ 3. choose remaining cards.

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ways

Observe

$$C(n,r) = C(n, n-r)$$

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!\,\underbrace{(n-(n-r))!}_{=r!}}$$

Counting Binary Sequences.

of binary strings of length n that contain exactly r 1's

$$= \binom{n}{r}$$

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How many distinguishable rearrangements are there?

length = 11

$\boxed{\begin{matrix} \binom{n}{n} = 1 \\ \binom{n}{0} = 1 \end{matrix}}$ *

$$\underline{}\ \underline{}\ \underline{I}\ \underline{}\ \underline{I}\ \underline{M}\ \underline{}\ \underline{}\ \underline{I}\ \underline{}\ \underline{I}$$

1 2 3 4 ... 11

$1\ M \to \binom{11}{1}$

$4\ Is \to \binom{10}{4}$

$4\ Ss \to \binom{6}{4}$

$2Ps \to \binom{2}{2} = 1$

$$\#\text{ ways} = \binom{11}{1}\binom{10}{4}\binom{6}{4}$$

$$\overbrace{(x+y)(x+y)\cdots\cdots(x+y)}^{n}$$

$$= \sum_{i=0}^{n} c_i x^i y^{n-i}$$

$i=0 \Rightarrow c_0 x^0 y^n \rightarrow c_0 = 1$

$i=n \rightarrow c_n x^n y^0 \rightarrow c_n = 1$

$$= c_0 x^0 y^n + c_1 x^1 y^{n-1} + c_2 x^2 y^{n-2}$$

$$+ \dots + \quad c_n x^n y^0$$

<u>Claim</u> $c_i = \binom{n}{i}$

<u>Binomial</u> <u>Theorem</u>

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

e.g.
$$(x+y)^2 = (x+y)(x+y)$$
$$= x^2 + 2xy + y^2$$
$$= \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$$
$$\binom{2}{0} = 1, \quad \binom{2}{1} = 2, \quad \binom{2}{2} = 1$$

$(x+y)^0$					1				
$(x+y)^1$				1		1			
$(x+y)^2$			1		2		1		
$(x+y)^3$		1		3		3		1	
$(x+y)^4$	1		4		6		4		1

Pascal's Triangle.

$$\boxed{\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}}$$

Counting Bags / Multisets

\# of ways to make an unordered list of r objects from $\{1, 2, \ldots n\}$ where repeats are allowed.

Theorem

\# of bags of length r taken from a set of size n with repetition allowed

= \# of solns to $x_1 + x_2 + \ldots + x_n = r$ using nonnegative integers

= \# of binary sequences of length $r + n - 1$ with exactly r 0's. $= \binom{r+n-1}{r}$

r 0's

$S \quad \boxed{0\,0}\,1\,\boxed{0}\,1\,\boxed{\ }\,1\,\boxed{0\,0\,0}\,1\,\boxed{0\,0}\,1\,\boxed{0\,0}\,1\,\boxed{0}\,1\,\boxed{0}$

$(n-1)$ 1's

$\Rightarrow$ n red boxes

let x_i = size of box i

So $x_1 + x_2 + \dots + x_n = r$

total length $S = r + (n-1)$

we choose r positions in S to be 0.

ways to do this

$$= \binom{r+(n-1)}{r}$$

Ex

3 flavors of jelly beans

pick any 10

$$\# \text{ of ways} = \binom{10+3-1}{10}$$

$$= \binom{12}{10}$$