

Propositional Logic

Defn A proposition is a statement that is either true or false.

P "This is CS 222" T

Q "My age is 19" F

A propositional variable stands for a prop.

P = true,

Q = false.

Logical Operators

AND, OR, NOT

Symbol form: \wedge \vee \neg

Ex $(P \wedge Q)$

↑
"Compound"
proposition.

($P \vee Q, \neg P \wedge Q, \dots$

allow us to build
more complex propositions
from simple ones.

Truth Tables

Ex. $P \vee \neg q$

| P | q | q $\neg q$ | $P \vee \neg q$ |
|---|---|-----------------------|-----------------|
| F | F | T | T |
| F | T | F | F |
| T | F | T | T |
| T | T | F | T |

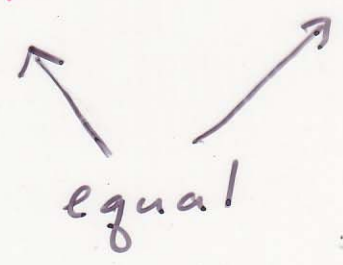
\oplus = exclusive - OR

| P | q | $P \oplus q \equiv (p \vee q) \wedge \neg (p \wedge q)$ |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | F |

Defn Two logical propositions are equivalent if and only if they have the same truth values in each row of a truth table. (\equiv)

Ex $\neg(p \vee q) \equiv \neg p \wedge \neg q$

| P | q | $\neg(p \vee q)$ | $\neg p \wedge \neg q$ |
|---|---|------------------|------------------------|
| F | F | T | T |
| F | T | F | F |
| T | F | F | F |
| T | T | F | F |



$$\neg (p \vee q) \equiv \neg p \wedge \neg q$$

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↑ De Morgan's laws *

Ex

"distributive property"

1. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
2. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

| P | q | r | $p \vee (q \wedge r)$ | $(p \vee q) \wedge (p \vee r)$ |
|---|---|---|-----------------------|--------------------------------|
| F | F | F | F | F |
| F | F | T | F | F |
| F | T | F | F | F |
| F | T | T | T | T |
| T | F | F | T | T |
| T | F | T | T | T |
| T | T | F | T | T |
| T | T | T | T | T |

equal, so \equiv

Ex Another knight / knave problem...

John says "If Bill is a knave then I am a knight"

[we can write this logically as]
 $\neg b \rightarrow j \quad (\equiv b \vee j)$
↑ will discuss more

Bill says "We are different"

[$(j \wedge \neg b) \vee (\neg j \wedge b)$]

Observation!

Either John is a knight and made a true statement
OR John is a knave and lied

so,

$(j \wedge (b \vee j)) \vee (\neg j \wedge \neg (b \vee j))$

(corrected version)

$$\begin{aligned}
& (j \wedge (b \vee j)) \vee (\neg j \wedge \neg(b \vee j)) \\
\equiv & (j \wedge (b \vee j)) \vee (\neg j \wedge (\neg b \wedge \neg j)) \\
\equiv & (j \wedge (b \vee j)) \vee (\neg b \wedge \neg j) \\
\equiv & (j \wedge (b \vee j)) \vee \neg(b \vee j) \\
\equiv & (j \vee \neg(b \vee j)) \wedge ((b \vee j) \vee \neg(b \vee j)) \\
\equiv & (j \vee \neg(b \vee j)) \wedge \top \\
\equiv & (j \vee \neg(b \vee j)) \\
\equiv & (j \vee (\neg b \wedge \neg j)) \\
\equiv & (j \vee \neg b) \wedge (j \vee \neg j) \\
\equiv & (j \vee \neg b) \wedge \top \\
\equiv & \boxed{j \vee \neg b}
\end{aligned}$$

this must be true
regardless of whether
John is telling the
truth or lying.

Observation 2

Bill is either a knight and his statement is true or a knave and his statement is false.

So,

$$\begin{aligned}
 & (b \wedge [(j \wedge \neg b) \vee (\neg j \wedge b)]) \\
 & \vee (\neg b \wedge \neg [(j \wedge \neg b) \vee (\neg j \wedge b)]) \\
 \equiv & ((\underbrace{b \wedge (j \wedge \neg b)}_F) \vee (b \wedge (\neg j \wedge b))) \\
 & \vee (\neg b \wedge \neg (j \wedge \neg b) \wedge \neg (\neg j \wedge b)) \\
 \equiv & (b \wedge \neg j) \vee (\neg b \wedge (\neg j \vee b) \wedge \neg (b \wedge \neg j)) \\
 \equiv & (b \wedge \neg j) \vee ((\neg b \wedge \neg j) \vee (\underbrace{\neg b \wedge b}_F)) \wedge \neg (b \wedge \neg j) \\
 \equiv & (b \wedge \neg j) \vee ((\neg b \wedge \neg j) \wedge \neg (b \wedge \neg j)) \\
 \equiv & ((b \wedge \neg j) \vee (\neg b \wedge \neg j)) \wedge (\underbrace{(b \wedge \neg j) \vee \neg (b \wedge \neg j)}_T) \\
 \equiv & (b \wedge \neg j) \vee (\neg b \wedge \neg j) \\
 \equiv & (\underbrace{b \vee \neg b}_T) \wedge \neg j \\
 \equiv & \boxed{\neg j} \text{ is true}
 \end{aligned}$$

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We can now combine
Obs. 1 and Obs. 2:

$$\begin{aligned} & (j \vee \neg b) \wedge \neg j \\ \equiv & (\underbrace{j \wedge \neg j}_F) \vee (\neg b \wedge \neg j) \\ \equiv & \boxed{\neg b \wedge \neg j} \end{aligned}$$

This is the answer:
Both John and Bill
are knaves.

[OK: Truth tables are
easier! ☺]