CS 222 Lec 6

Proofs by Mathematical Induction

$\forall n > 0$, $\boxed{1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}}$

$P(n)$

<u>Defn</u> A <u>statement</u> <u>about</u> <u>the</u> <u>positive</u> <u>integers</u> is a predicate $P(n)$ whose domain is the positive integers ($\mathbb{Z}^{>0}$).

$\boxed{\forall n > 0, P(n)}$ *

* - would like to prove.

(Weak)

"Principle of Mathematical Induction"

Let $P(n)$ be a statement about the positive integers.

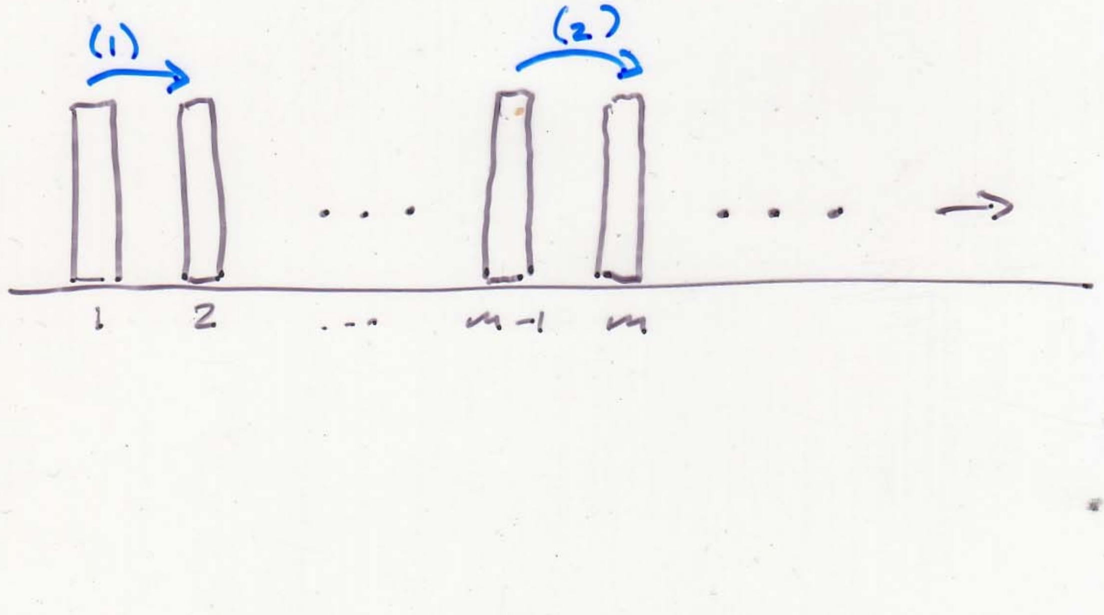
If :

basis (1) $P(1)$ is true, and

inductive step (2) $\forall\, m \geq 2,\ P(m-1) \rightarrow P(m)$

Then '$\forall n > 0\ P(n)$' is true.

"like dominoes"



Example

Let $P(n)$: $1 + 2 + \dots + n = \frac{n(n+1)}{2}$

Show $\forall n > 0,\ P(n)$

Proof

By math. induction on n

Need to show

step 1 (basis) $P(1)$ is true:

lhs = left hand side of equation

$lhs = 1$

$rhs = \frac{1 \cdot (1+1)}{2} = \frac{2}{2} = 1$ ✓

So $P(1)$ is true.

step 2 (inductive step)

let $m \geq 2$.

Assume $P(m-1)$ is true.

$$1 + 2 + \dots + (m-1) = \frac{(m-1)m}{2}$$

Next, we want to show $P(m)$ is true

P(m):

$$lhs = 1 + 2 + \dots + (m-1) + m$$

$$= \frac{(m-1)m}{2} + m$$

by inductive assumption that $P(m-1)$ is true.

$$= \frac{(m-1)m + 2m}{2}$$

$$= \frac{m^2 + m}{2} = \frac{m(m+1)}{2}$$

$$= rhs.$$

So $P(m-1) \rightarrow P(m)$

So $P(n)$ is true for $n > 0$.

"Strong" Version of Math. Induction

Same as weak, except line 2 is:

(2) $\forall m \geq 2 \quad P(1), P(2), \dots P(m-1) \rightarrow P(m)$

Example

Let $x \in \mathbb{R}$ (real numbers)
[Let $|x| < 1$]

$$P(n): \quad 1 + x + x^2 + \dots + x^n = \frac{1-x^{n+1}}{1-x}$$

Claim: $\forall n \geq 1, \quad P(n)$

Proof by induction on n

step 1 (basis) Show $P(1)$ is true.

$$lhs = 1 + x$$

$$rhs = \frac{1-x^2}{1-x} = \frac{(1-x)(1+x)}{(1-x)}$$

$$= 1 + x \checkmark$$

step 2 (inductive step)

Show $P(m-1) \rightarrow P(m)$

Assume $P(m-1)$:

$$1 + x + \dots + x^{(m-1)} = \frac{1-x^m}{1-x}$$

P(m):

$$\text{lhs} = 1 + x + \dots + x^{(m-1)} + x^m$$

$$= \frac{1-x^m}{1-x} + x^m$$

(by ind. assump.)

$$= \frac{1-x^m + x^m(1-x)}{1-x}$$

$$= \frac{1-x^{m+1}}{1-x} = \text{rhs.}$$

By induction $P(n)$ is true for all $n \geq 1$.

Application. Let $|x| < 1$

$$1 + x + x^2 + x^3 + \cdots$$

$$\lim_{n \to \infty} \left(1 + x + x^2 + \cdots + x^n \right) = ?$$

$$= \lim_{n \to \infty} \left(\frac{1 - x^{n+1}}{1 - x} \right) = \frac{1}{1 - x}$$

e.g. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$$= \frac{1}{1 - \left(\frac{1}{2}\right)} = 2$$