CS 222 Lec 8

Pigeohole Principle

(1) $n+1$ pigeons

n holes

$\exists$ a hole with ≥ 2 pigeons

"Generalized"

(2) $m \cdot n + 1$ pigeons

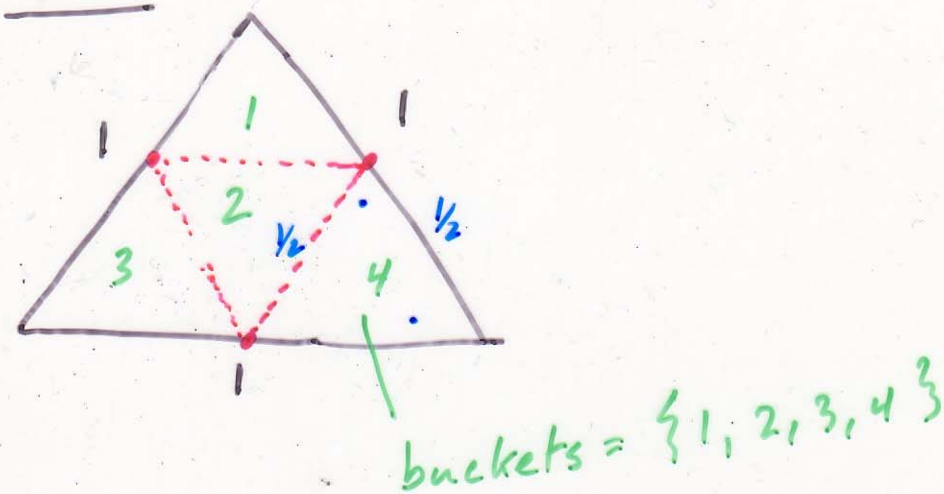
n holes

$\exists$ a hole with $\geq m+1$ pigeons

Example

Claim: Every equalateral triangle (sidelength = 1) contain 5 pts, has two points that are within a distance of $1/2$ from each other.

Proof



There are 4 buckets and 5 points, So one bucket must contain two points → these at most a distance of 1/2 apart.

□

<u>Claim</u> Given any 7 integers a subset of 4 of them will have the sum of their squares divisible by 4.

$$\{a, b, c, d, e, f, g\}$$

$$\Downarrow$$

$\{a, c, d, f\}$ some subset

Claim says $4 \mid (a^2 + c^2 + d^2 f^2)$

<u>Proof</u> use Pigeonhole Principle

pigeons = given 7 numbers

$\boxed{x}$ $\qquad$ $\boxed{y}$

$4 \mid x^2$ $\qquad$ $4 \nmid y^2$

pigeonholes

By G.P.H.P., one of these buckets has ≥ 4 numbers

<u>Case 1:</u> $\lfloor x \rfloor$ bucket has ≥ 4 numbers
$4 \mid x^2$

choose 4 of these numbers for the subset x_1, x_2, x_3, x_4

$$4 \mid (x_1^2 + x_2^2 + x_3^2 + x_4^2)$$

<u>Case 2:</u> $\lfloor y \rfloor$ bucket has ≥ 4 numbers
$4 \nmid y^2$

for each number y $4 \nmid y^2$

so $2 \nmid y \Rightarrow y$ is odd

$$y = 2k + 1$$

$$y^2 = (2k+1)^2 = 4k^2 + 4k + 1$$

$$y^2 = 4m + 1$$

pick y_1, y_2, y_3, y_4 from the bucket

$$y_1^2 = 4m_1^2 + 1$$

$$y_2^2 = 4m_2^2 + 1$$

$$\vdots$$

$$(y_1^2 + y_2^2 + y_3^2 + y_4^2) = 4(\quad) + 4$$

so this is divisible by 4

$\square$

Sets

Defn a set is a collection of elements

e.g. $\mathbb{R}$ = set of real numbers

C = people in this class (finite)

$x \in A$ = x is an element of A

$\mathbb{Z}$ = set of integers

$\mathbb{Q}$ = rational numbers

$\mathbb{N}$ = natural numbers $= \{0, 1, 2, \ldots\}$

Subsets

Sets A, B

A is a subset of B if

$$\forall x \in A \Rightarrow x \in B$$

notation $A \subseteq B$

e.g. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

Equality

$$A = B \Leftrightarrow \begin{array}{l} 1) \; A \subseteq B \\ 2) \; B \subseteq A \end{array}$$

$\mathbb{Q} \subsetneq \mathbb{R}$ since

$\sqrt{2} \in \mathbb{R}$ but $\sqrt{2} \notin \mathbb{Q}$

Empty Set $\emptyset = \{\}$

Note $\emptyset \subseteq A$ for all sets A

Set-Builder Notation

$$A = \{ x \in \mathbb{Z} \mid 3x + 2 \geq 10 \}$$

all ~~the~~ elements must satisfy all conditions listed

$$B = \{ 5y + 6 \mid y \in \mathbb{N} \}$$

Set Operations

Intersection

$$A \cap B = \{ x \mid x \in A \ \underline{\text{and}} \ x \in B \}$$

Union

$$A \cup B = \{ x \mid x \in A \ \underline{\text{or}} \ x \in B \}$$

Set-Difference

$$A - B = \{ x \mid x \in A \text{ and } x \notin B \}$$

<u>Defn</u> Sets $A + B$ are <u>disjoint</u>

if $A \cap B = \emptyset$

e.g. $\mathbb{Z}^{<0} \cap \mathbb{N} = \emptyset$

<u>Complement</u> (need a universal set of reference U)

$$\bar{A} = A^c = \{ x \in U \mid x \notin A \}$$

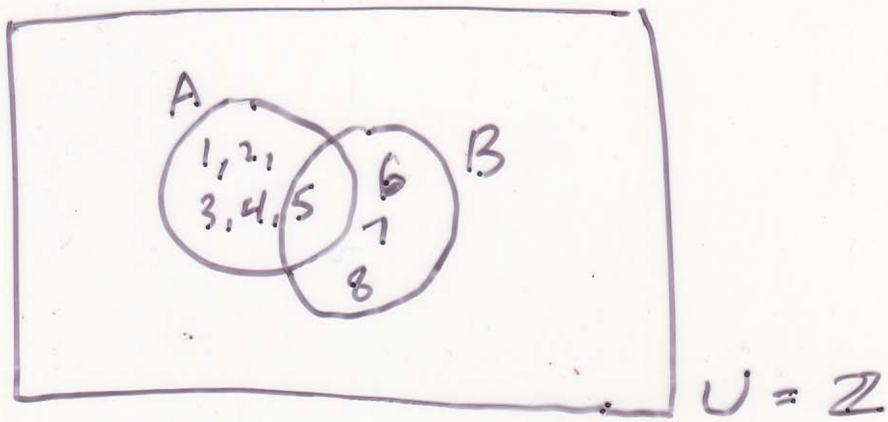
$= A'$

$U = \mathbb{Z}$

$A =$ ~~$\{\ldots\}$~~ $\{$ even numbers $\}$

$\bar{A} =$ odd numbers

Venn Diagrams



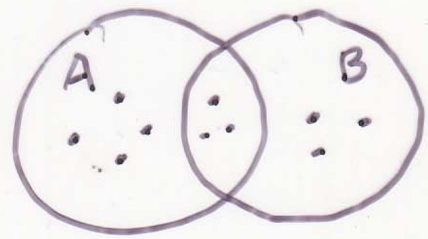
$$A - B = A \cap \overline{B}$$

Defn for finite sets, the size of a set A is the number of elements in it.

notation $|A|$ = size of A

Principle of Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- \left(|A \cap B| + |B \cap C| + |A \cap C| \right)$$
$$+ |A \cap B \cap C|$$