Review Problems

3.1, ex 3

$$\{x \in \mathbb{Z} \mid x = 2y,\ y \in \mathbb{Z}\}$$

$$\{x \in \mathbb{N} \mid x = 2^y,\ y \in \mathbb{N}\}$$

$$O = \{x \in \mathbb{N} \mid x = 2 \cdot y + 1,\ y \in \mathbb{N}\}$$

$$\{x \in \mathbb{N} \mid x = y^2 \text{ and } y \in O\}$$

3.1, 13

a) If $A \subseteq B$ then $n(A) \leq n(B)$

TRUE

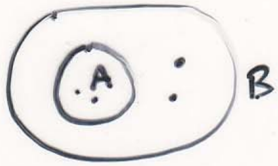
b) If $n(A) \leq n(B)$ then $A \subseteq B$

FALSE

$A = \{1, 2\}$ $\quad B = \{3, 4, 5\}$

c) If $A \subseteq B$ and $A \neq B$

Then $n(A) < n(B)$



d) If $n(A) < n(B)$ then $A \subseteq B$

FALSE

$A = \{1, 2\}$ $B = \{3, 4, 5\}$

3.2, 11

a) $A \subseteq B \Rightarrow A \times C \subseteq B \times C$

TRUE

b) $(A \cup B) \times (A - B) = A^2 - B^2$

$A = \{1, 2\}$

$B = \{2, 3\}$

$A \cup B = \{1, 2, 3\}$

$A - B = \{1\}$

$$(A \cup B) \times (A - B) = \{ (1,1), (2,1), (3,1) \}$$

$$A^2 - B^2 = (A \times A) - (B \times B)$$

$$= \{ (1,1), (1,2), (2,1), (2,2) \}$$

$$- \{ (2,2), (2,3), (3,2), (3,3) \}$$

$$= \{ (1,1), (1,2), (2,1) \}$$

$\neq$

FALSE

c) $A \times (B \times C) = (A \times B) \times C$

FALSE

$A = \{a\} \quad B = \{b\} \quad C = \{c\}$

$$B \times C = \{ (b, c) \}$$

$$A \times (B \times C) = \{ (a, (b, c)) \}$$

$$(A \times B) \times C = \{ ((a, b), c) \}$$

$\neq$

3.2, 12

Which is larger $\mathcal{P}(A \times B)$

or $\mathcal{P}(A) \times \mathcal{P}(B)$?

$$|\mathcal{P}(A)| = 2^{|A|}$$

$$|\mathcal{P}(B)| = 2^{|B|}$$

$$\boxed{|\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^{|A| \cdot |B|}}$$

$$|\mathcal{P}(A) \times \mathcal{P}(B)| = |\mathcal{P}(A)| \cdot |\mathcal{P}(B)|$$

$$= 2^{|A|} \cdot 2^{|B|} = 2^{|A| + |B|}$$

this is typically larger.

$A = \{a\}$ $B = \{b\}$

3.3, 11 b

Give element-wise proofs

that b) $A \cap B = A \Rightarrow A \cup B = B$

Proof

1) $A \cup B \subseteq B$

let $x \in A \cup B. \Rightarrow x \in A$ or $x \in B$

suppose $x \in A$, then $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B \Rightarrow x \in B$.

2) $B \subseteq A \cup B$

let $x \in B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in A \cup B$.

d) $A \cup B = B,\ B \cup C = C \Rightarrow A \cup C = C$

Proof

1) $A \cup C \subseteq C$

let $x \in A \cup C$. then $x \in A$ or $x \in C$ ✓

suppose $x \in A$. then $x \in A$ or $x \in B$, $x \in A \cup B$

then $x \in B$, then $x \in B$ or $x \in C$, so $x \in B \cup C$

$\Rightarrow x \in C$

2) $C \subseteq A \cup C$

let $x \in C$. then $x \in C$ or $x \in A$

$\Rightarrow x \in A \cup C$.

3.4 4b Show $ab + bc = (a+c)b$

$ab + bc = ab + cb$ commutative

$= (a+c)b$ distributative.

4d Show $ab + (a' + c)' = a(b+c')$

$ab + (a'+c)' = ab + a'' \cdot c'$ DeMorgan

$= ab + a \cdot c'$ Idempotent

$= a(b + c')$ Distrib.

3.4, 6b

If $ab = a$ then $a' + b = 1$

$$
\begin{aligned}
a' + b &= (ab)' + b \\
&= (a' + b') + b \\
&= a' + (b' + b) \\
&= a' + 1 \\
&= 1
\end{aligned}
$$

6d

If $ab' = 0$ then $a + b = b$

$$ab' = 0$$

$$ab' + b = b$$

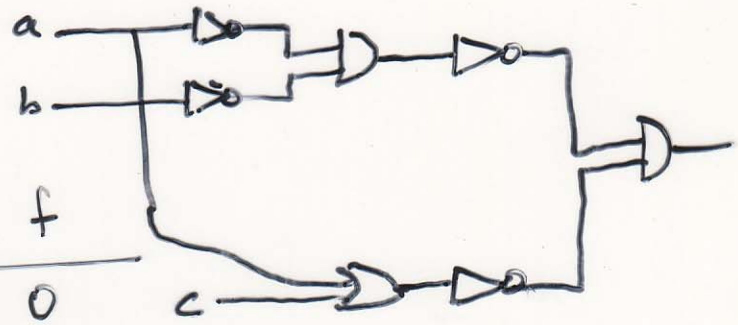
$$(a+b)\cdot(b'+b) = \cancel{+}\, b$$

$$(a+b)\cdot 1 = b$$

$$\Rightarrow \quad a + b = b$$

3.5 3d

$$f(a,b,c) = (a' \cdot b')' \cdot (a + c)'$$



truth table

a	b	c	f
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

3.5, 11 b

$f = xyz + xy'z' + xy'z + x'yz + x'y'z$

	~~x~~ yz	$y'z$	$y'z'$	yz'
x	1	1	1	0
x'	1	1	0	0

$f = z + xy'$

Prove $b_k = 3b_{k-1} + 2$

$b_1 = 2$

$\downarrow$

$b_n = 3^n - 1$ for all $n \geq 1$

Proof by induction.

base case $(n=1)$

$b_1 = 2$ by defn

$b_1 = 3^1 - 1 = 2$

agree ✓

ind. step

assume $b_k = 3^k - 1$

by rec. defn $b_{k+1} = 3 \cdot b_k + 2$

So $b_{k+1} = 3(3^k - 1) + 2$

by inductive hyp.

$= 3^{k+1} - 3 + 2$

$= 3^{k+1} - 1$ ✓

by induction, claim proved.