

# Chapter 1. Logic and Proofs

## 1.1 Propositional Logic

Proposition  $\equiv$  a declarative sentence that is either true or false.

Ex. Seattle is the capital city of Washington State. (F)  
The solution of  $x^2 - 1 = 0$  is  $x = 1$ . (F)

Negation ( $\neg$ , NOT, -)

Ex. Today is Monday. Negation: Today is not Monday.

Conjunction ( $\wedge$ , AND,  $\cdot$ )

Disjunction ( $\vee$ , OR, +)

Exclusive-or ( $\oplus$ , XOR,  $\oplus$ )

Truth table

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$\neg p$
T	T	T	T	F	F
T	F	F	T	T	F
F	T	F	T	T	T
F	F	F	F	F	T

Implication (Conditional Statement)

$p \rightarrow q$   
(hypothesis) (conclusion)

if p, then q  
p implies q  
p only if q  
⋮

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex. If a cat can fly, a whale can run.

Converse, Inverse, Contrapositive

$p \rightarrow q$	$q \rightarrow p$ (converse)
$\neg p \rightarrow \neg q$ (inverse)	$\neg q \rightarrow \neg p$ (contrapositive)

Note.  $p \rightarrow q \equiv \neg q \rightarrow \neg p$

**Biconditional ( $\leftrightarrow$ )**

$$p \leftrightarrow q \equiv (p \rightarrow q) \text{ and } (q \rightarrow p)$$

Ex. You can fly an airplane iff you buy a ticket.

**Precedence**

(high)  $\neg$   
 $\wedge$   
 $\vee$   
 $\rightarrow$   
 (low)  $\leftrightarrow$

**Translating English sentences**

Ex. You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.

$q \equiv$  You can ride a roller coaster.  
 $r \equiv$  You are under 4 feet tall.  
 $s \equiv$  You are older than 16 years old.

$$(r \wedge \neg s) \rightarrow \neg q \quad \text{or} \quad q \rightarrow \neg r \vee s$$

**Logic Puzzles**

Ex:

Knights - truth  
 Knaves - lie *unprincipled, crafty person*

A: "B is a knight."

B: "The two of us are opposite type."

What are A and B?

**Solution**

Case 1: A is a knight

A tells a truth and B is a knight

What B said is a lie  $\rightarrow$  contradiction

Case 2: A is a knave.

A tells a lie  $\rightarrow$  B is a knave.

What B said is a lie  $\rightarrow$  consistent

Answer: A and B are knaves.

Sudoku (数独)**Conceptis Sudoku**

by Dave Green

	7	9	4		5		3	
4			1		7			6
1				3				
2	3						4	8
		1		2		6		
6	9						1	7
				1				9
3			7		4			5
	6		8		2	3	7	

**Difficulty Level** ★

5/15



## Logic and Bit Operation

$1 \leftrightarrow \text{true}$   
 $0 \leftrightarrow \text{false}$   
 AND  $\leftrightarrow \wedge$   
 OR  $\leftrightarrow \vee$   
 XOR  $\leftrightarrow \oplus$

Ex.  $\begin{array}{r} 1011 \ 0110 \\ 0001 \ 1101 \\ \hline 1011 \ 1111 \quad \text{bitwise OR} \\ 0001 \ 0100 \quad \text{bitwise AND} \\ 1010 \ 1011 \quad \text{bitwise XOR} \end{array}$

## 1.2 Propositional Equivalences

Tautology  $\rightarrow$  always true      Ex.  $p \vee \neg p$   
 Contradiction  $\rightarrow$  always false      Ex.  $p \wedge \neg p$

### Logical Equivalences

$p \equiv q \quad (p \leftrightarrow q \text{ is a tautology.})$   
 $\Leftrightarrow$

Ex. Show that  $p \rightarrow q$  and  $\neg p \vee q$  are logically equivalent.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Ex.  $p \rightarrow q \equiv \neg q \rightarrow \neg p$  (Contrapositive)

p	q	$p \rightarrow q$	$\neg q$	$\neg p$	$\neg q \rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Ex.  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$  (Distributive law)

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Dual

$\vee \rightarrow \wedge$   
 $\wedge \rightarrow \vee$   
 $T \rightarrow F$   
 $F \rightarrow T$

Ex.  $\neg(p \wedge q) \equiv \neg p \vee \neg q$  DeMorgan

Dual:  $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Ex. Distributive law

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Dual:  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

<b>TABLE 6 Logical Equivalences.</b>	
<i>Equivalence</i>	<i>Name</i>
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

<b>TABLE 7 Logical Equivalences Involving Implications.</b>
$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

<b>TABLE 8 Logical Equivalences Involving Biconditionals.</b>
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Ex 5.  $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

(Method 1)

$$\begin{aligned}
 \neg(p \vee (\neg p \wedge q)) &= \neg p \wedge \neg(\neg p \wedge q) && \text{DeMorgan} \\
 &= \neg p \wedge (\neg \neg p \vee \neg q) && \text{DeMorgan} \\
 &= \neg p \wedge (p \vee \neg q) && \text{double negation} \\
 &= (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{distributive} \\
 &= F \vee (\neg p \wedge \neg q) \\
 &= (\neg p \wedge \neg q) \vee F && \text{commutative} \\
 &= (\neg p \wedge \neg q) && \text{identity}
 \end{aligned}$$

(Method 2)

p	q	$\neg p$	$\neg p \wedge q$	$p \vee (\neg p \wedge q)$	$\neg(p \vee (\neg p \wedge q))$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F	F
T	F	F	F	T	F	T	F
F	T	T	T	T	F	F	F
F	F	T	F	F	T	T	T

Note. Boolean algebra notation

$$\begin{aligned}
 &\overline{p + \bar{p} \cdot q} && \bar{p} \cdot \bar{q} \\
 &= \bar{p} \cdot (\overline{\bar{p} \cdot q}) \\
 &= \bar{p} \cdot (\bar{\bar{p}} + \bar{q}) \\
 &= \bar{p} \cdot (p + \bar{q}) \\
 &= \bar{p} \cdot p + \bar{p} \cdot \bar{q} \\
 &= 0 + \bar{p} \cdot \bar{q} \\
 &= \bar{p} \cdot \bar{q}
 \end{aligned}$$

### 1.3 Predicates and Quantifiers

Ex.  $P(x)$ : " $x > 5$ " // propositional function  $P$  at  $x$  or predicate //

Ex. What is the truth value of  $P(4)$ ?

$P(4)$ : " $4 > 5$ "  $\rightarrow$  "false"

#### Quantifiers

##### (1) Universal Quantifier

$\forall x P(x)$  // for all  $x$ ,  $P(x)$  //

Ex.  $P(x)$ : " $x + 1 > x$ " where  $x$  is real number

$\forall x P(x)$  is true

Ex 7. What is the truth value of  $\forall x P(x)$ , where  $P(x)$  is " $x^2 < 10$ " and the domain consists of the positive integers not exceeding 4?

$\forall x P(x) \rightarrow P(1) \wedge P(2) \wedge P(3) \wedge P(4)$

$P(4)$ : " $4^2 < 10$ " is false. Therefore  $\forall x P(x)$  is false.

Note. To prove  $\forall x P(x)$  is false, we need one value that  $P(x)$  is false. Such a value of  $x$  is called "counterexample."

Ex 10.  $P(x)$ :  $x^2 > 0$  for all integer  $x$ .

Counterexample.  $x = 0$

##### (2) Existential Quantifier

$\exists x P(x)$  // There exists an element  $x$  in the domain such that  $P(x)$  is true //

Ex.  $P(x)$ :  $x^2 > 10$  where  $x$  is positive integer  $\leq 4$

Domain: {1, 2, 3, 4}

$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3) \vee P(4)$

True

	When true?	When false?
$\forall x P(x)$	P(x) is true for every x	There is an x for which P(x) is false
$\exists x P(x)$	There is an x for which P(x) is true	P(x) is false for every x

### Binding Variables

Ex.  $\exists x Q(x,y)$  // x is bound, y is free //

### Negation

(1) "Every student took CS 160":  $\forall x P(x)$

Negation: "There is a student who did not take CS 160"

$$\exists x \neg P(x)$$

$$\therefore \boxed{\neg \forall x P(x) \equiv \exists x \neg P(x)}$$

(2) "There is a student who took CS 330":  $\exists x Q(x)$

Negation: "No student took CS 330"

"It is not the case that there is a student who took CS 330"

$$\forall x \neg Q(x)$$

$$\therefore \boxed{\neg \exists x Q(x) \equiv \forall x \neg Q(x)}$$

Ex : What is the negation of the statement, "There is an honest politician".

$$\exists x P(x)$$

Negation:

$$\neg \exists x P(x) \equiv \forall x \neg P(x) \quad // \text{Every politician is dishonest} //$$

Translating from English into logical expressions

Ex : "Some student in this class has visited Mexico"

(Method 1)

→ "There is a student in this class with property that the student has visited Mexico"

→ "There is a student x in this class having property x has visited Mexico"

x

M(x)

→  $\exists x M(x)$

(Method 2)

"There is a person x having the property that x is a student in this class and

x

S(x)

x has visited Mexico

M(x)

$$\exists x (S(x) \wedge M(x))$$

**LOGIC PROGRAMMING**

PROLOG (CS 355)

**Ex.**

**instructor(P,C)  
enrolled(S,C)**

**Fact:**

**instructor(chan,math181)  
instructor(patel,cs222)  
instructor(grossman,ee261)  
enrolled(Alfred,math181)  
enrolled(brian,cs222)  
enrolled(carol,ee261)  
enrolled(carol,math181)  
enrolled(david,cs222)**

**Rule:**

**teaches(P,S) :- instructor(P,C),enrolled(S,C)**

**prolog**

-----  
**?enrolled(brian,cs222)  
yes**

-----  
**?enrolled(X,cs222)  
brian  
david**

-----  
**?teaches(X,carol)  
grossman  
chan**

<head> :- <body>

rule stmt  
(condition)

```

parent(X,Y) :- father(X,Y).
parent(X,Y) :- mother(X,Y).
grandparent(X,Z) :- parent(X,Y), parent(Y,Z).
ancestor(X,Z) :- parent(X,Z).
ancestor(X,Z) :- parent(X,Y), ancestor(Y,Z).
sibling(X,Y) :- mother(M,X), mother(M,Y),
                father(F,X), father(F,Y), X \= Y.
cousin(X,Y) :- parent(U,X), parent(V,Y), sibling(U,V).

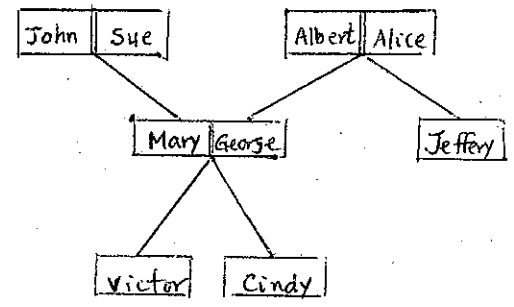
```

fact stmt  
(hypothesis)

```

father(albert, jeffrey).
mother(alice, jeffrey).
father(albert, george).
mother(alice, george).
father(john, mary).
mother(sue, mary).
father(george, cindy).
mother(mary, cindy).
father(george, victor).
mother(mary, victor).

```



Family Tree

Note: if <head> is omitted, the clause is considered a goal.

goal stmt  
(goal)  
(query)

```

:- ancestor(john, cindy).
yes

```

```

:- ancestor(X, cindy), sibling(X, jefferey).
X = george ;
no

```

```

:- sibling(A, B).
A = jefferey, B = george ;
A = george, B = jefferey ;
A = cindy, B = victor ;
A = victor, B = cindy ;
no

```

## 1.4 Nested Quantifiers

Ex.  $\forall x \forall y (x + y = y + x)$  // commutative law //

$\forall x \exists y (x + y = 0)$  //  $y = -x$  : additive inverse //

Ex. Translate the statement into English.

$$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$$

“Product of a positive number and a negative number is negative”

Ex. Translate the statement “The sum of two positive integers is positive” into a logical expression.

$$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y) > 0)$$

### Negating Nested Quantifiers

Ex. Negation of  $\forall x \exists y (xy = 1)$

$$\neg \forall x \exists y (xy = 1)$$

$$\exists x \neg \exists y (xy = 1)$$

$$\exists x \forall y \neg (xy = 1)$$

$$\exists x \forall y (xy \neq 1)$$

\*Does order matter?

Ex. 4.  $Q(x,y) \equiv “x + y = 0”$

(1)  $\exists y \forall x Q(x,y)$  // false //

(2)  $\forall x \exists y Q(x,y)$  // true //

## 1.5 Rules of Inference

<b>TABLE 1 Rules of Inference.</b>		
<i>Rule of Inference</i>	<i>Tautology</i>	<i>Name</i>
$\frac{p}{\therefore p \vee q}$	$p \rightarrow (p \vee q)$	Addition
$\frac{p \wedge q}{\therefore p}$	$(p \wedge q) \rightarrow p$	Simplification
$\frac{p}{\therefore p \wedge q}$ $\frac{q}{\therefore p \wedge q}$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	Conjunction
$\frac{p}{\therefore q}$ $\frac{p \rightarrow q}{\therefore q}$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus ponens ①
$\frac{\neg q}{\therefore \neg p}$ $\frac{p \rightarrow q}{\therefore \neg p}$	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
$\frac{p \rightarrow q}{\therefore p \rightarrow r}$ $\frac{q \rightarrow r}{\therefore p \rightarrow r}$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$\frac{p \vee q}{\therefore q}$ $\frac{\neg p}{\therefore q}$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Disjunctive syllogism
$\frac{p \vee q}{\therefore q \vee r}$ $\frac{\neg p \vee r}{\therefore q \vee r}$	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution

① If the weather tomorrow is good, I will go fishing.

P

Q

$$(P \wedge (P \rightarrow Q)) \rightarrow Q$$

or

$$\frac{P}{P \rightarrow Q}$$

$$\therefore Q$$

## Using Rules of Inference to Build Arguments

**Ex 6.** Show that the hypotheses “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”

**Solution:**

Let  $p \equiv$  “It is sunny this afternoon”  
 $q \equiv$  “It is colder than yesterday”  
 $r \equiv$  “We will go swimming”  
 $s \equiv$  “We will take a canoe trip”  
 $t \equiv$  “We will be home by sunset”

<u>Then</u>	<u>Hypothesis</u>	<u>Conclusion</u>
	$\neg p \wedge q$	$t$
	$r \rightarrow p$	
	$\neg r \rightarrow s$	
	$s \rightarrow t$	

<u>Step</u>	<u>Reason</u>
1. $\neg p \wedge q$	Hypothesis
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Hypothesis
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Hypothesis
6. $s$	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Hypothesis
8. $t$	Modus ponens using (6) and (7)

## Resolution

$$((p \vee q) \wedge (\neg p \vee r)) \xrightarrow{\text{resolvent}} (q \vee r)$$

Ex 8.

$$\left( \underbrace{\text{Jasmine is skiing}}_q \text{ or } \underbrace{\text{it is not snowing}}_p \right)$$

$$\text{and } \left( \underbrace{\text{It is snowing}}_{\neg p} \text{ or } \underbrace{\text{Bart is playing hockey}}_r \right)$$

$$\text{implies } \underbrace{\text{Jasmine is skiing}}_q \text{ or } \underbrace{\text{Bart is playing hockey}}_r$$

1.  $(q \vee p) \wedge (\neg p \vee r)$       given
2.  $(p \vee q) \wedge (\neg p \vee r)$       commutative
3.  $(q \vee r)$       resolution

## Fallacies

*A statement or argument that is based on false or invalid inference.*

Ex 10. If you do every problem in this book, then you will

$\overset{p}{\text{learn discrete math.}} \quad \text{You learned discrete math.}$

$\underset{q}{\text{Therefore, you did every problem in this book.}}$

$$[(p \rightarrow q) \wedge q] \rightarrow p ?$$

P	q	$P \rightarrow q$	$(P \rightarrow q) \wedge q$	$[(P \rightarrow q) \wedge q] \rightarrow P$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Not tautology

## Rules of Inference for Quantified Statements

### 1. Universal instantiation

We show  $P(c)$  is true where  $c$  is a particular member, given  $\forall x P(x)$

Ex.  $\forall x P(x)$  : All women are wise

“Lisa is wise” // Lisa is a particular member of //  
// the domain of all women //

### 2. Universal generalization

We show  $\forall x P(x)$  is true, given the premise that  $P(c)$  is true for all members  $c$  in the domain.

Note. We take arbitrary element  $c$  and show that  $P(c)$  is true.

### 3. Existential instantiation

We conclude that there is an element  $c$  in the domain for which  $P(c)$  is true if we know that  $\exists x P(x)$  is true.

### 4. Existential generalization

We conclude that  $\exists x P(x)$  is true when a particular element  $c$  with  $P(c)$  true is known.

<i>Rule of Inference</i>	<i>Name</i>
$\frac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

Ex 12. Show that the premises, “Everyone in this class has taken CS 160” and “Lisa is a student in this class” imply “Lisa has taken CS 160”.

**Solution:**

$A(x) \equiv$  “x is in this class”

$B(x) \equiv$  “x has taken CS 160”

**Premises:**  $\forall x (A(x) \rightarrow B(x))$   
 $A(\text{Lisa})$

**Conclusion:**  $B(\text{Lisa})$

1.  $\forall x (A(x) \rightarrow B(x))$             given
2.  $A(\text{Lisa}) \rightarrow B(\text{Lisa})$         universal instantiation
3.  $A(\text{Lisa})$                         given
4.  $B(\text{Lisa})$                         modus ponens for 2 and 3

## EXAMPLE 13

Show that the premises “A student in this class has not read the book,” and “Everyone in this class passed the first exam” imply the conclusion “Someone who passed the first exam has not read the book.”

*Solution:* Let  $C(x)$  be “ $x$  is in this class,”  $B(x)$  be “ $x$  has read the book,” and  $P(x)$  be “ $x$  passed the first exam.” The premises are  $\exists x(C(x) \wedge \neg B(x))$  and  $\forall x(C(x) \rightarrow P(x))$ . The conclusion is  $\exists x(P(x) \wedge \neg B(x))$ . These steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8) ◀

**Remark:** Mathematical arguments often include steps where both a rule of inference for propositions and a rule of inference for quantifiers are used. For example, universal instantiation and modus ponens are often used together. When these rules of inference are combined, the hypothesis  $\forall x(P(x) \rightarrow Q(x))$  and  $P(c)$ , where  $c$  is a member of the universe of discourse, show that the conclusion  $Q(c)$  is true.

## Paradox

[1]

"All Cretans are liars", said a philosopher in Crete.

[2]

- The next bulleted statement is a lie.
- The previous bulleted statement is true.

[3] **Russel's Paradox**

Proposition:

There is a barber who lives on an island. The barber shaves all those men who live on the island who do not shave themselves, and only those men.

Question:

Does the barber shave himself?

Answer:

If the barber shaves himself then he is a man on the island who shaves himself hence he, the barber, does not shave himself. If the barber does not shave himself then he is a man on the island who does not shave himself hence he, the barber, shaves him(self).

Solution

1. different type
- 2.

## Methods of Proving Theorems

### (1) Direct proof

**Ex.1** Prove that "If  $n$  is odd integer, then  $n^2$  is odd integer"

**Proof:**

$$n \text{ is odd} \rightarrow n = 2k + 1$$

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = \underbrace{2k(2k + 2)}_{\text{odd}} + 1 \quad \text{q.e.d.}$$

**Ex. 7.** Prove that the sum of two rational numbers is rational.

$$* \text{ rational number} = p / q \quad (q \neq 0)$$

**Proof:**

$$r = p / q \quad (q \neq 0)$$

$$s = t / u \quad (u \neq 0)$$

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + tq}{qu} \quad \underbrace{\hspace{1.5cm}}_{\text{rational}} \quad \text{q.e.d.}$$

**(2) Indirect proofs**

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Note Proof by contrapositive

**Ex. 3.** Prove that "If  $3n+2$  is odd, then  $n$  is odd"

**Proof:** Suppose  $n$  is even. Then  $n = 2k$ .

$$3n + 2 = 3(2k) + 2 = 6k + 2 = \underbrace{2(3k + 1)}_{\text{even}}$$

**Ex. 8.** Prove that if  $n^2$  is odd, then  $n$  is odd.

**Proof:** Suppose  $n$  is even. Then  $n = 2k$ .

$$n^2 = (2k)^2 = 4k^2 = \underbrace{2(2k^2)}_{\text{even}}$$

**(3) Proof by Contradiction**

**Ex 10.** Prove that  $\sqrt{2}$  is irrational.

**Proof:**

Suppose  $\sqrt{2}$  is rational.

## Exhaustive Proof

Ex. Prove that  $(n + 1)^3 \geq 3^n$ ,  $n$  is positive integer with  $n \leq 4$ .

Proof.

	Left	>	Right	
n = 1:	$(1 + 1)^3 = 8$	>	$3 = 3^1$	
n = 2:	$(2 + 1)^3 = 27$	>	$9 = 3^2$	
n = 3:	$(3 + 1)^3 = 64$	>	$27 = 3^3$	
n = 4:	$(4 + 1)^3 = 125$	>	$81 = 3^4$	q.e.d.
n = 5:	$(5 + 1)^3 = 216$	<	$243 = 3^5$	

Ex. Show that there are no solutions in integers  $x$  and  $y$  of  $x^2 + 3y^2 = 8$ .

Proof.

$$0 \leq x^2 \leq 8 \quad \text{and} \quad 0 \leq y^2 \leq 2$$

$$\text{Possible values: } x = -2, -1, 0, 1, 2 \quad y = -1, 0, 1$$

$$\text{Largest value for: } x^2 = 4 \quad y^2 = 1$$

$$\text{Largest value for } x^2 + 3y^2 = 7$$

There is no solution for  $x^2 + 3y^2 = 8$  where  $x$  and  $y$  are integers.

## Proof by Cases

Ex. Show that  $|x \cdot y| = |x| \cdot |y|$ ,  $x, y$  real numbers

Proof.

There are only four cases:

- Case 1:  $x \geq 0, y \geq 0$ :  $|x \cdot y| = xy = |x| \cdot |y|$   
 Case 2:  $x \geq 0, y < 0$ :  $|x \cdot y| = -xy = x(-y) = |x| \cdot |y|$   
 Case 3:  $x < 0, y \geq 0$ :  $|x \cdot y| = -xy = (-x)y = |x| \cdot |y|$   
 Case 4:  $x < 0, y < 0$ :  $|x \cdot y| = xy = (-x)(-y) = |x| \cdot |y|$       q.e.d.

## Mistakes in Proof

Ex.  $2 = 1$

Proof.

Let $x = 1$	
$x^2 = x$	multiply by $x$
$x^2 - 1 = x - 1$	subtract 1
$(x+1)(x-1) = x - 1$	factoring
$x + 1 = 1$	divide by $(x-1)$
$1 + 1 = 1$	substitute $x = 1$
$2 = 1$	q.e.d. (?)