11. Boolean Algebra

George Boole (1854)

Triple [K,•, +] consisting of a finite set of elements K, subject to an equivalence relation "=" and two binary operators, "+" and "•", such that for every element x and y ∈ K, the operations x+y and x•y are uniquely defined and the following postulates are satisfied.

1.	The operations are closed for all x and y ∈ K.	x+y ∈ K and x•y ∈ K	
2.	Commutative laws	$A \cdot B = B \cdot A$	$A + B = B + A$
3.	Distributive laws	$A \cdot (B+C) = (A \cdot B) + (A \cdot C)$	$A + (B \cdot C) = (A+B) \cdot (A+C)$
4.	Associative laws	$A \cdot (B \cdot C) = (A \cdot B) \cdot C$	$A + (B+C) = (A+B) + C$
5.	∃ Identity elements	$1 \cdot A = A$	$0 + A = A$
6.	Inverse elements	$A \cdot \bar{A} = 0$	$A + \bar{A} = 1$

"Duality"

Note: When K = {0,1}, it is called 'switching algebra'.

- **Logic variables - true (1) or false (0)**
- **operations**

AND	OR	NOT
(∩	∪	~)
(·	+	−)

Boolean expression (Boolean function)

1. **0,1,x,$\bar{x}$ are Boolean expression.**
2. **If A and B are Boolean expression, then $\bar{A}$,$(\bar{A})$, A+B, A•B are also Boolean expression**

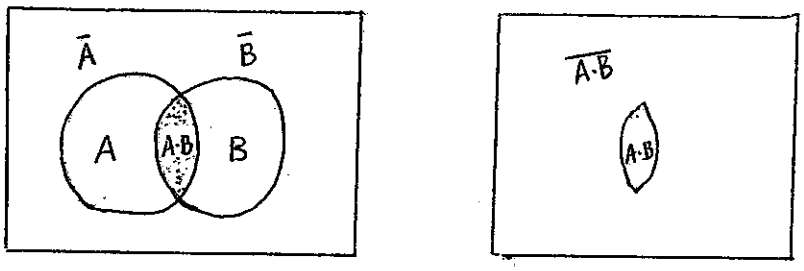
Boolean Operators

P	Q	$\bar{P}$	P•Q	P+Q	P xor Q	P nand Q	P nor Q
0	0	1	0	0	0	1	1
0	1	1	0	1	1	1	0
1	0	0	0	1	1	1	0
1	1	0	1	1	0	0	0

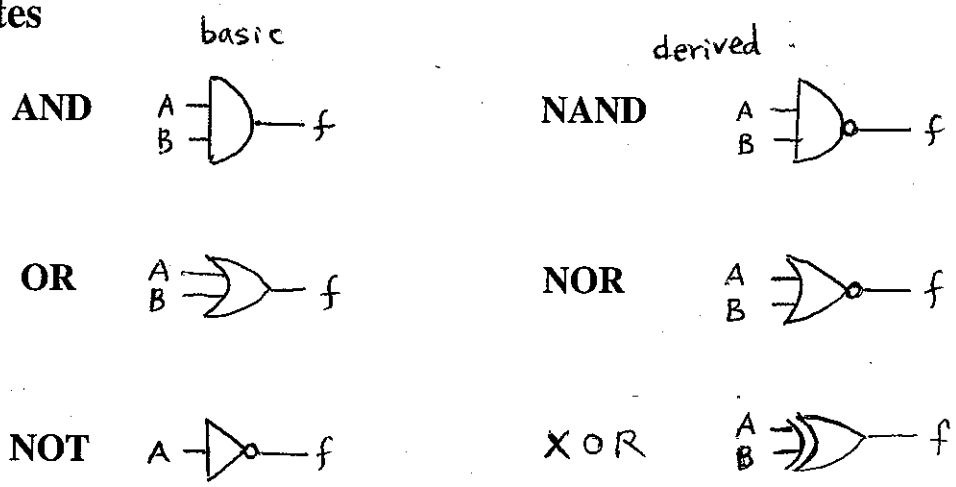
DeMorgan's Theorem

$$\overline{A \cdot B} = \overline{A} + \overline{B} \qquad \overline{A + B} = \overline{A} \cdot \overline{B}$$

Venn Diagram



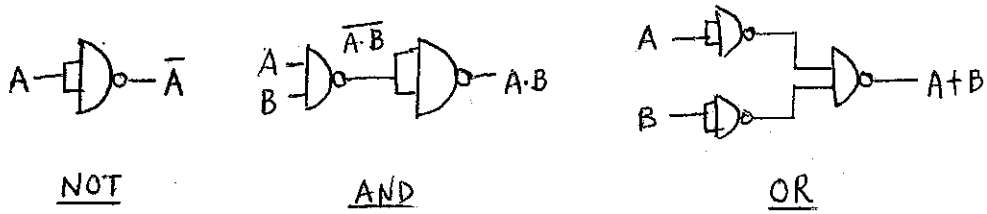
Gates



Functionally complete set of gates

1. AND, OR, NOT
2. AND, NOT
3. OR, NOT
4. NAND
5. NOR

$$A + B = \overline{\overline{A + B}} = \overline{\overline{A} \cdot \overline{B}}$$



Combinational Circuit — output depends on input only

(Sequential Circuit) — output depends on input and past history of input (memory)

Implementation of Boolean Functions

Ex.

	A	B	C	F
0	0	0	0	0
1	0	0	1	0
2	0	1	0	1 ✓
3	0	1	1	1 ✓
4	1	0	0	0
5	1	0	1	0
6	1	1	0	1 ✓
7	1	1	1	0

$\bigvee_{i=0}^{7}$ $F(A,B,C) = \sum(2,3,6)$ // sigma notation //

↕ dual

$\bigwedge_{i=0}^{7}$ $F(A,B,C) = \prod(0,1,4,5,7)$ // pi notation //

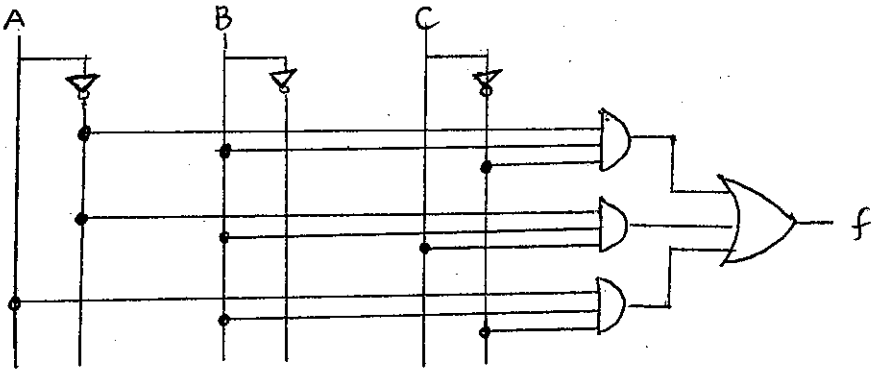
characteristic number $= (01001100)_2$

$= 76$

Canonical Expression

(1) SOP (sum-of-product) form

$$f = \bar{A}\cdot B\cdot\bar{C} + \bar{A}\cdot B\cdot C + A\cdot B\cdot\bar{C}$$



(2) POS (product-of-sum) form

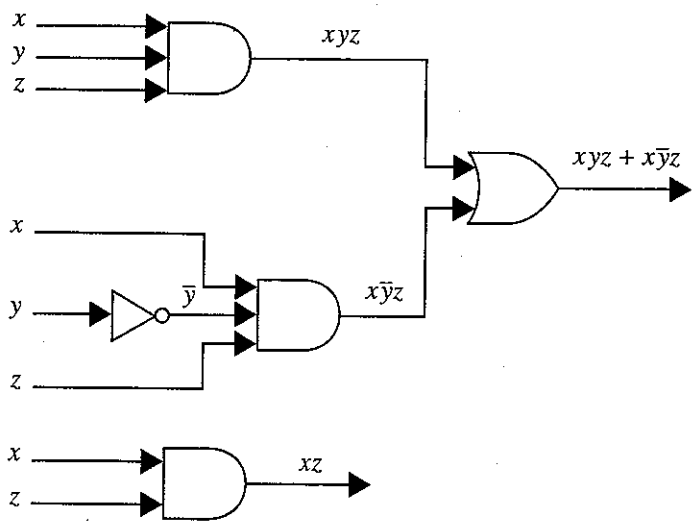
$$f = (\overline{\bar{A}\,\bar{B}\,\bar{C}})\cdot(\overline{\bar{A}\,\bar{B}\cdot C})\ (\overline{A\,\bar{B}\,\bar{C}})\ (\overline{A\,\bar{B}\,C})\ (\overline{A\,B\,C})$$

$$= (A+B+C)\ (A+B+\bar{C})\ (\bar{A}+B+C)\ (\bar{A}+B+\bar{C})\ (\bar{A}+\bar{B}+\bar{C})$$

"Equivalent"

11.4 Minimization of Circuit

Motivation



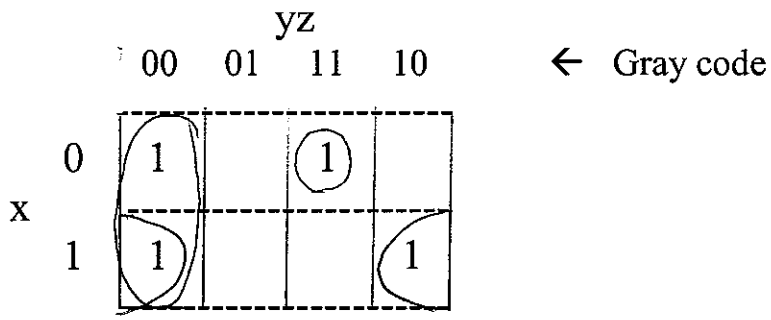
Note. $xyz + x\bar{y}z = xz(y + \bar{y}) = xz$

- Algebraic method
- Quine-McCluskey method

- Karnaugh Map (K-map) method

K-map Method

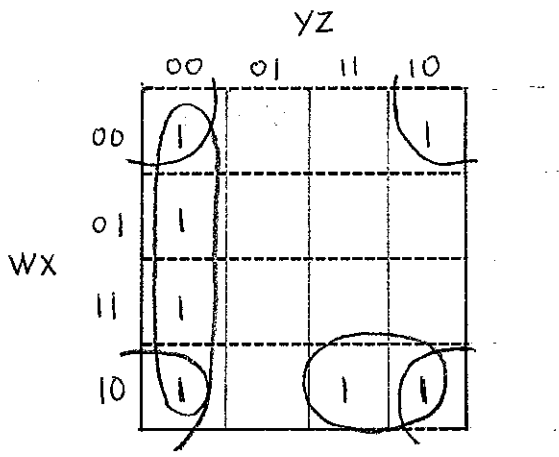
Ex. 3 (a) $f(x,y,z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}\bar{z}$
$\qquad\qquad\qquad\quad 110 \quad 100 \quad 011 \quad 000$



$f(x,y,z) = x\bar{z} + \bar{y}\bar{z} + \bar{x}yz$

Ex. 4 (b)

$f(w,x,y,z) = wx\bar{y}\bar{z} + w\bar{x}yz + w\bar{x}y\bar{z} + w\bar{x}\bar{y}\bar{z} + \bar{w}x\bar{y}\bar{z} + \bar{w}\bar{x}yz + \bar{w}\bar{x}\bar{y}\bar{z}$
$\qquad\qquad\quad 1100 \quad 1011 \quad 1010 \quad 1000 \quad 0100 \quad 0010 \quad 0000$

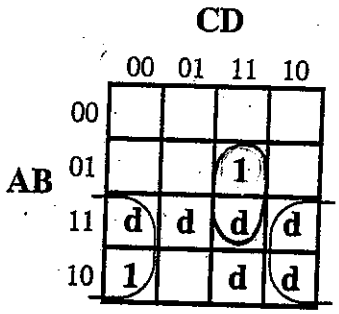


$f(w,x,y,z) = \bar{y}\bar{z} + \bar{x}\bar{z} + w\bar{x}y$

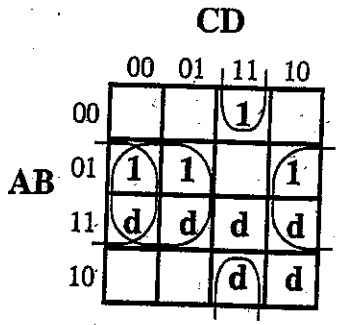
Ex.

Table Truth Table for the One-Digit Packed Decimal Incrementer

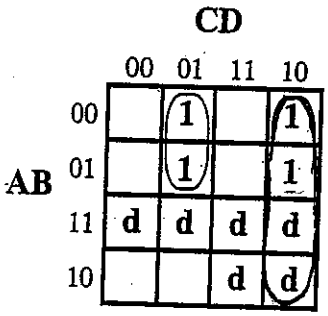
	Input					Output			
Number	A	B	C	D	Number	W	X	Y	Z
0	0	0	0	0	1	0	0	0	1
1	0	0	0	1	2	0	0	1	0
2	0	0	1	0	3	0	0	1	1
3	0	0	1	1	4	0	1	0	0
4	0	1	0	0	5	0	1	0	1
5	0	1	0	1	6	0	1	1	0
6	0	1	1	0	7	0	1	1	1
7	0	1	1	1	8	1	0	0	0
8	1	0	0	0	9	1	0	0	1
9	1	0	0	1	0	0	0	0	0
Don't care condition	1	0	1	0		d	d	d	d
	1	0	1	1		d	d	d	d
	1	1	0	0		d	d	d	d
	1	1	0	1		d	d	d	d
	1	1	1	0		d	d	d	d
	1	1	1	1		d	d	d	d



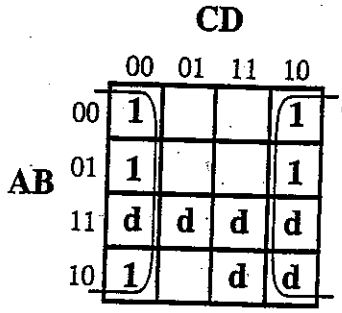
(a) $W = A\bar{D} + BCD$



(b) $X = B\bar{D} + B\bar{C} + \bar{B}CD$



(c) $Y = \bar{A}\bar{C}D + C\bar{D}$



(d) $Z = \bar{D}$

Figure Karnaugh Maps for the Incrementer