Chapter 2. Sets, Functions, Sequence, Sum

2.1 Sets

Def. unordered collection of objects

Georg Cantor
Bertrand Russel

Notation

$N = \{0, 1, 2, 3, \ldots\}$
$Z = \{\ldots, -1, 0, 1, 2, \ldots\}$
$Z^+ = \{1, 2, 3, \ldots\}$
R: real number

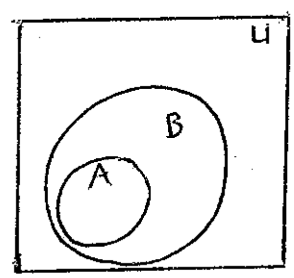
Def. Two sets are equal iff they have the same elements.

Note. empty set = ϕ or $\{\}$

$\emptyset \neq \{\emptyset\}$
Singleton set

Def. $A \subseteq B$ // A is a subset of B //

$\forall x\ (x \in A \rightarrow x \in B)$



$A \subsetneq B$ proper subset

Thm 1. For any set S,

(i) $\emptyset \subseteq S$
(ii) $S \subseteq S$

Def. finite set

$|S|$ = number of elements in S (cardinality)

infinite set

Cartesian Product

- ordered n-tuple
  $(a_1, a_2, \ldots, a_n)$

Note. 2-tuple ≡ ordered pair

Def. Cartesian product

$$A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}$$

Ex. $A = \{1, 2\}$, $B = \{a, b, c\}$

$$A \times B = \{ (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) \}$$

Def. <u>Relation</u> is a subset R of the Cartesian product from set A to set B.

Ex. $R_1 = \{ (1,a), (2,b), (2,c) \}$

Power set

$P(S) \equiv$ set of all subsets

Ex. $A = \{a, b\}$

$P(A) = \{ \emptyset, \{a\}, \{b\}, \{a, b\}\}$

$|A| = 2, \ |P(A)| = 2^2 = 4$

Set operations

- **Union**

  $A \cup B = \{x \mid x \in A \text{ or } x \in B \}$

- **Intersection**

  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

  Disjoint: $A \cap B = \emptyset$

- **Difference (relative complement)**

  $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

- **Symmetric difference**

  $A \triangle B = \{x \mid (x \in A \text{ or } x \in B) \text{ and } (x \notin A \cap B)\}$

  $= \{x \mid x \in A \cup B \text{ and } x \notin A \cap B\}$

Ex. $A = \{1, 3, 5\}, B = \{4, 5, 6\}, U = \{1, 2, 3, 4, 5, 6\}$

$A \cup B = \{1, 3, 4, 5, 6\}$

$A \cap B = \{5\}$

$A - B = \{1, 3\}$

$B - A = \{4, 6\}$

$A \triangle B = \{1, 3, 4, 6\}$

$\bar{A} = \{2, 4, 6\}$ // complement //

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Ex 10

Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$

($\rightarrow$) $x \in \overline{A \cup B} \rightarrow x \notin A \cup B$

$\rightarrow x \notin A$ and $x \notin B$

$\rightarrow x \in \overline{A}$ and $x \in \overline{B}$

$\rightarrow x \in \overline{A} \cap \overline{B}$

So $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

($\leftarrow$) $x \in \overline{A} \cap \overline{B} \rightarrow x \in \overline{A}$ and $x \in \overline{B}$

$\rightarrow x \notin A$ and $x \notin B$

$\rightarrow x \notin A \cup B$

$\rightarrow x \in \overline{A \cup B}$

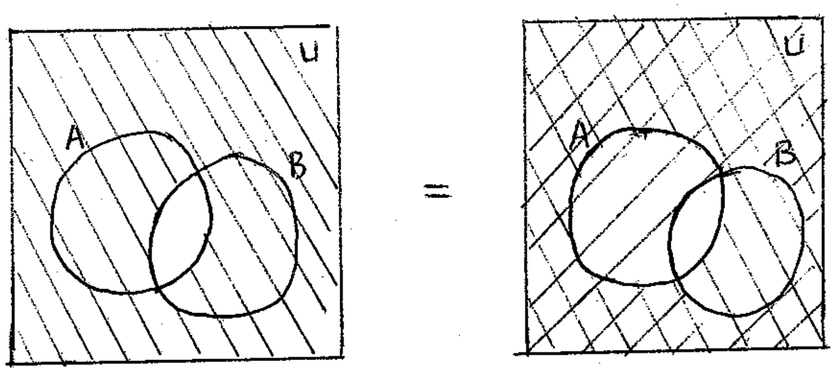
Therefore, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

We showed $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B} \rightarrow \overline{A \cup B} = \overline{A} \cap \overline{B}$

Venn Diagram

- **John Venn (1834 – 1923)**

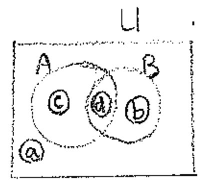
Ex. $\overline{A \cap B} = \overline{A} \cup \overline{B}$ **(DeMorgan's law)**



Membership table

Ex. $A, B \subseteq U$

An element $x \in U$ satisfies exactly one of the following 4 situations:



a) $x \notin A, \quad x \notin B$
b) $x \notin A, \quad x \in B$
c) $x \in A, \quad x \notin B$
d) $x \in A, \quad x \in B$

	A	B	$A \cap B$	$A \cup B$	$\overline{A}$
a)	0	0	0	0	1
b)	0	1	0	1	1
c)	1	0	0	1	0
d)	1	1	1	1	0

Note. We can establish the equality of two sets by comparing their respective columns in the table.

Ex. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

A	B	$A \cap B$	$\overline{A \cap B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cup \overline{B}$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

$\therefore \; \overline{A \cap B} = \overline{A} \cup \overline{B}$

Ex. Simplify: $\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$

$$
\begin{aligned}
\overline{\overline{(A \cup B) \cap C} \cup \overline{B}} &= \overline{\overline{((A \cup B) \cap C)}} \cap \overline{\overline{B}} && \text{DeMorgan} \\
&= ((A \cup B) \cap C) \cap B && \text{double complement} \\
&= (A \cup B) \cap (C \cap B) && \text{associative} \\
&= (A \cup B) \cap (B \cap C) && \text{commutative} \\
&= [\,(A \cup B) \cap B\,] \cap C && \text{associative} \\
&= B \cap C && \text{absorption}
\end{aligned}
$$

Note. $\neg[\neg[(p \vee q) \wedge r] \vee \neg q] \rightarrow q \wedge r$

Generalized Unions and Intersections

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots A_n$$

Computer Representation of Sets

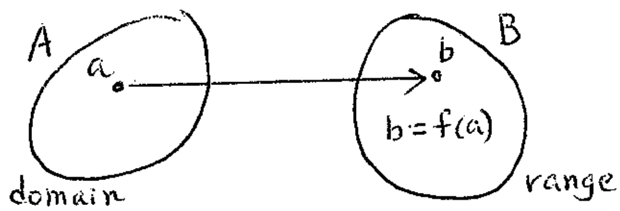
Ex U = {1, 2, 3, 4, 5, 6, 7, 8}

S = {1, 3, 5, 6}

1 0 1 0 1 1 0 0

2.3 Functions

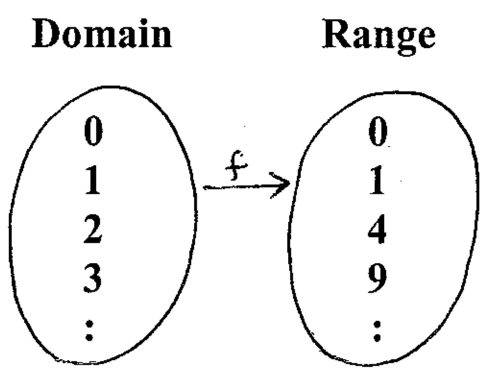
Def. **A function from set A to set B is an assignment of exactly one element of B to each element of A**



Note. A function is a special kind of relation. (Chapter 8)

Ex. $f: Z \rightarrow Z$

$f(x) = x^2$



Def. $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$

Ex. $f_1, f_2: R \rightarrow R$

$f_1(x) = x^2, \quad f_2(x) = x - x^2$

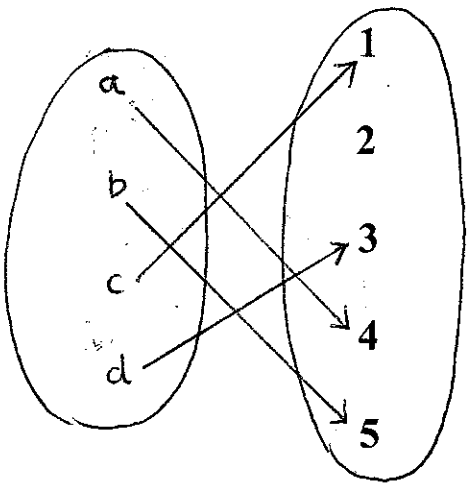
$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$

$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = x^2(x - x^2) = x^3 - x^4$

One-to-one (injective)

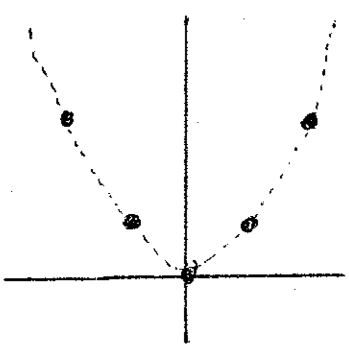
$f(x) \neq f(y)$ whenever $x \neq y$ For each $y \in Y$, there is at most one $x \in X$ with $f(x) = y$

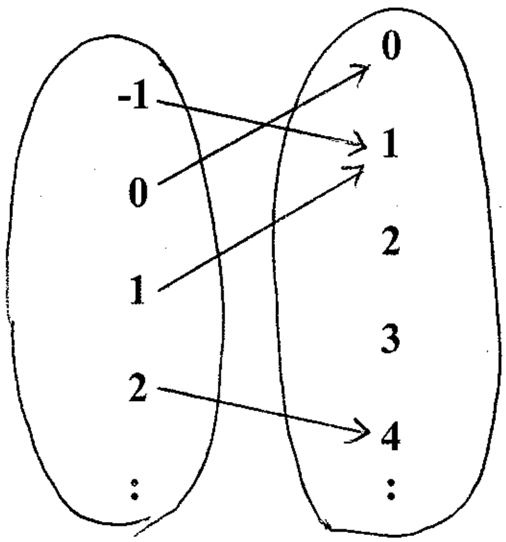
Ex. **Determine whether the following function is one-to-one.**



Ans: ______

Ex. **Determine whether $f(x) = x^2$ is one-to-one.**





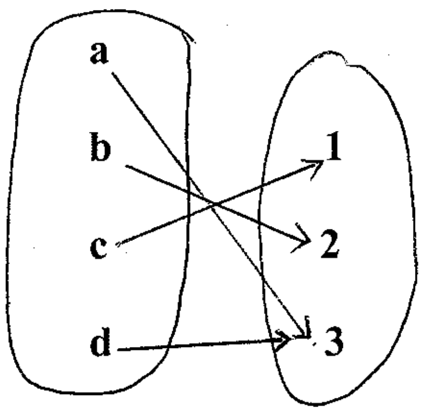
Ans: ______

Onto (surjective)

$\forall y \ \exists x \ (f(x) = y)$ range of f is Y

Note. Every y in the range has corresponding x in the domain.

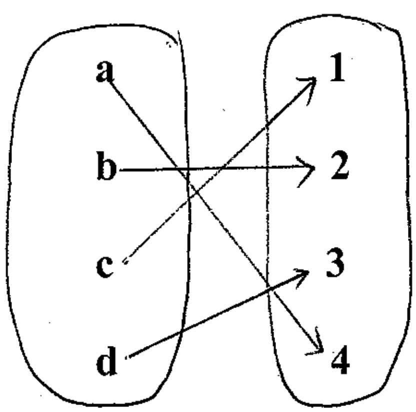
Ex. Determine whether the following function is onto.



Ans: __________

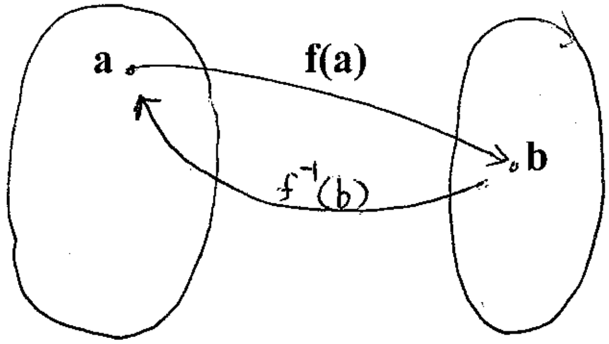
One-to-one and Onto (bijection)

Ex. Determine whether the following function is bijection.



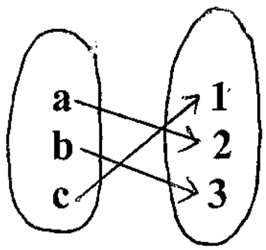
Ans: __________

Inverse Function



Note. One-to-one correspondence is invertible.

Ex.



$f(a) = 2$
$f(b) = 3$
$f(c) = 1$

$f^{-1}(1) = c$
$f^{-1}(2) = a$
$f^{-1}(3) = b$

Ex. $f(x) = x^2$ **Is f invertible? Ans:________**

Composition of function

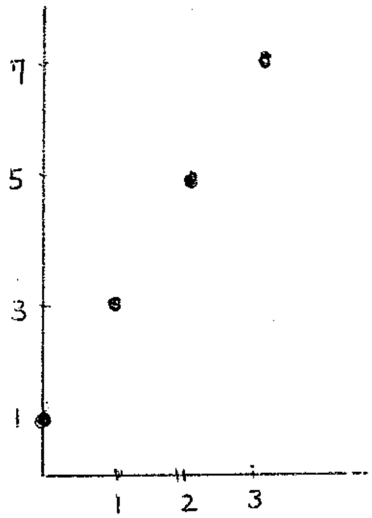
$$(f \circ g)(a) = f(g(a))$$

Ex. $f(x) = 2x + 3$
$g(x) = 3x + 2$

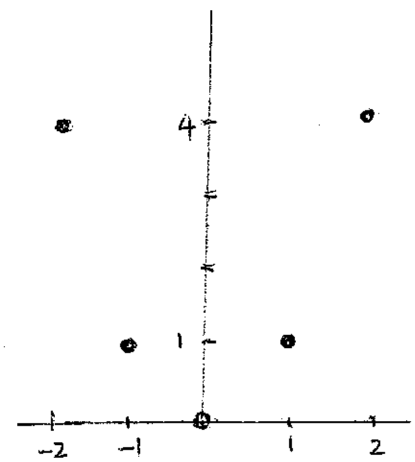
$$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2) + 3 = 6x + 7$$
$$(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3) + 2 = 6x + 11$$

Graph of Function

Ex. **N to N** — $f(n) = 2n + 1$



Z to Z — $f(n) = n^2$



Floor and Ceiling

Ex. $\lfloor 2.7 \rfloor = 2$ $\quad \lfloor -1/2 \rfloor = -1$ $\quad (\lfloor -x \rfloor = -\lceil x \rceil)$

$\lceil 2.7 \rceil = 3$ $\quad \lceil -1/2 \rceil = 0$ $\quad (\lceil -x \rceil = -\lfloor x \rfloor)$

Factorial Function

$f: N \rightarrow Z^+$

$f(3) = 3! = 1 \cdot 2 \cdot 3 = 6$

2.4 Sequences and Summations

Arithmetic sequence: a, a+d, a+2d, ..,, a+nd

Geometric progression: $a, ar, ar^2, \ldots ar^n$

Ex. 5 Find formulas for the following sequences with first five terms.

(a) 1, ½, ¼, 1/8, 1/16 $a_n = 1/2^{n-1}$

(b) 1, 3, 5, 7, 9 $a_n = 2n - 1$

(c) 1, -1, 1, -1, 1 $a_n = (-1)^{n-1}$ or $(-1)^{n+1}$

Ex. 6 1,2,2,3,3,3,4,4,4,4, 5,5,5,5,5,6,…

Ex. 7 5, 11, 17, 23, 29, 35, 41, 47, $a_n = 6n - 1$

Ex. 8 1, 7, 25, 79, 241, 727, 2185,

Hint. n-th term ≈ 3 * (n-1st term) → 3

Ans. $a_n = 3^n - 2$

Summations

Ex 10.

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

Thm 1. $$\sum_{j=0}^{n} a\cdot r^j = \begin{cases} \dfrac{ar^{n+1}-a}{r-1} & r\neq 1 \\ (n+1)a, & r=1 \end{cases}$$

(i) $r\neq 1$

$$S = a + ar + ar^2 + \cdots + ar^n \quad \cdots\cdots ①$$

$$) \; r\cdot S = \quad ar + ar^2 + \cdots + ar^n + ar^{n+1} \quad \cdots ②$$

$$②-① \quad (r-1)S = ar^{n+1} - a$$

$$\therefore S = \frac{ar^{n+1}-a}{r-1}$$

(ii) $r=1$ $$\sum_{j=0}^{n} a\cdot r^j = \overbrace{a + a + \cdots + a}^{n+1} = (n+1)a$$

Ex 13.

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6\cdot\sum_{i=1}^{4} i$$

$$= 6(1+2+3+4)$$

$$= 60$$

Ex 14. $\sum_{s\in\{0,2,4\}} s = 0+2+4 = 6$

Ex 15. $\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$

$= \frac{100\cdot 101\cdot 201}{6} - \frac{49\cdot 50\cdot 99}{6}$

$= 338{,}350 - 40{,}425$

$= 297{,}925$

Note. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Ex 16. $|x| < 1$. Find $\sum_{n=0}^{\infty} x^n$

$\sum_{n=0}^{k} x^n = \frac{x^{k+1}-1}{x-1}$

Because $|x|<1$, x^{k+1} approaches 0 as $k\to\infty$.

$\sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} \frac{x^{k+1}-1}{x-1} = \frac{-1}{x-1} = \frac{1}{1-x}$

Finite set vs. Infinite set

- Finite set

  Ex. $S = \{a, b, c, d\}$ $|S| = 4$
  $\emptyset = \{\ \}$ $|\emptyset| = 0$

- Infinite set

  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ Father
  $\mathbb{O} = \{1, 3, 5, 7, \dots\}$ Son
  $\mathbb{E} = \{0, 2, 4, 6, \dots\}$ Daughter

  Question
  $$|\mathbb{O}| \overset{\leq}{\underset{?}{=}} |\mathbb{N}|$$

Def. Two sets A and B are ***equinumerous*** if there is a bijection $f: A \rightarrow B$. ($f^{-1}: B \rightarrow A$)

Ex. A : multiple of 17 $\{0, 17, 34, 51, \dots\}$
B : perfect squares $\{0, 1, 4, 9, \dots\}$

$$f(17n) = n^2$$

Note. Countably Infinite $\equiv$ equinumerous with $\mathbb{N}$

Countable vs. Uncountable

countable - finite or countably infinite
uncountable

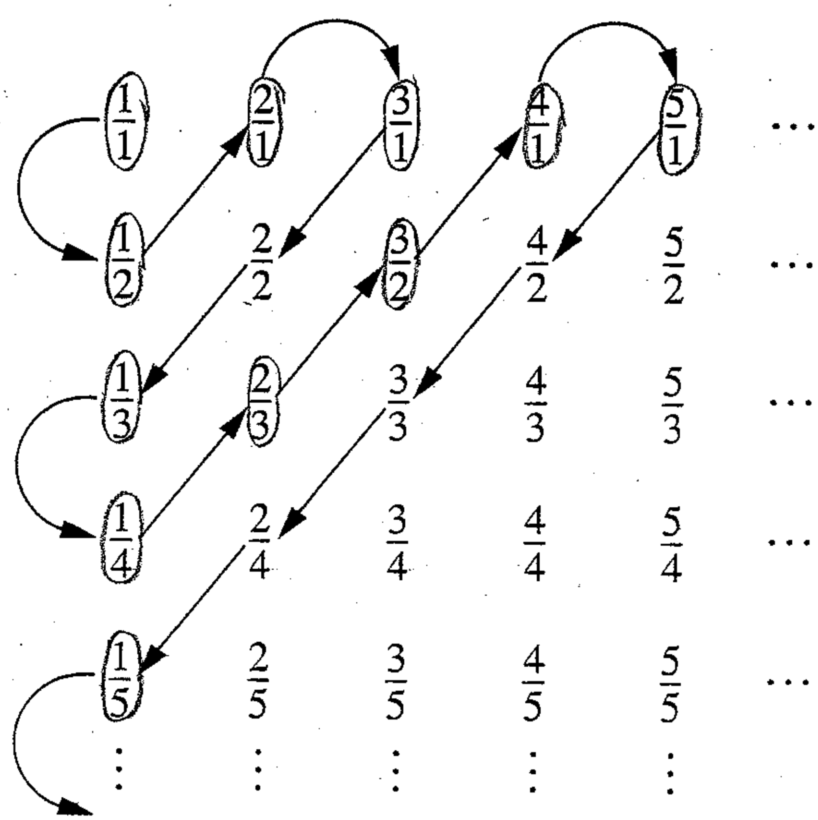
Ex 18. Set of odd positive integers are countable.

n	f(n)
1	1
2	3
3	5
4	7
:	:

$$f(n) = 2n - 1$$

one-to-one correspondence

Ex 20. Set of positive rational numbers is countable.



Ex 21. Set of real numbers is uncountable.

Suppose the set of real numbers is countable.
Then the real numbers between 0 and 1 is countable.
Therefore, all the real numbers between 0 and 1 can be listed as follow.

$$
\begin{array}{l}
0.\,d_{11}\ d_{12}\ d_{13}\ d_{14}\ d_{15}\ \ldots \\
0.\,d_{21}\ d_{22}\ d_{23}\ d_{24}\ d_{25}\ \ldots \\
0.\,d_{31}\ d_{32}\ d_{33}\ d_{34}\ d_{35}\ \ldots \\
0.\,d_{41}\ d_{42}\ d_{43}\ d_{44}\ d_{45}\ \ldots \\
\quad \vdots
\end{array}
\qquad d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}
$$

$r = 0.\,\bar{d}_{11}\ \bar{d}_{22}\ \bar{d}_{33}\ \bar{d}_{44}\ \bar{d}_{55} \ldots$

Let's form a new real number r as follow. Take the first digit from the 1st row, and change it to it's 9's complement. Then take the second digit from the 2nd row, and change it to its 9's complement, and so on.

0 ⟷ 9 1 ⟷ 8 2 ⟷ 7 3 ⟷ 6 4 ⟷ 5

We claim that r is not listed on the original table.

Why? Because r differs at least one digit from **any row.**

We assumed we listed all the real numbers, but we find new real number r. → contradiction
Therefore, set of real numbers is uncountable.

Note. Cantor's diagonalization argument
Principle

Georg Cantor (1845 – 1918)

$|\mathbb{N}| = \aleph_0$ ← smallest cardinality of infinite sets

$|\mathbb{R}| = \aleph_1$

$\vdots$

$\aleph_0 < \aleph_1 < \aleph_2 \ldots$ infinite

$A \quad 2^A \quad (2^A)^A$

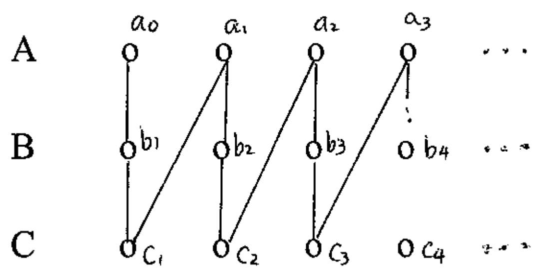
Ex. $A = \{a_0, a_1, a_2, \cdots\}$

$B = \{b_0, b_1, b_2, \cdots\}$

$C = \{c_0, c_1, c_2, \cdots\}$

$|A \cup B \cup C| = \aleph_0$

Proof (by dovetailing)



Ex. $\mathbb{N} \times \mathbb{N}$?