Chapter 5 Counting

5.1 Basics of Counting

(1) Product Rule.

Ex 2 **Numbering with letter (A-Z) and integer (1-100)**

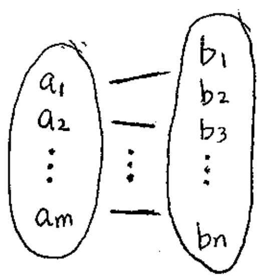
Ans. $26 \times 100 = 2600$

Ex 5 **License plate:** $\underbrace{\square\ \square\ \square}_{\text{letters}}\ \underbrace{\square\ \square\ \square}_{\text{digits}}$

Ans. $26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17{,}576{,}000$

Ex 7 **one-to-one function**

$m \leq n$



	choice
a_1	n
a_2	$(n-1)$
a_3	$(n-2)$
$\vdots$	$\vdots$
a_m	$(m-n+1)$

Ans. <u>$n(n-1)(n-2)\ldots(n-m+1)$</u>

Ex 9. k:= 0
for i_1:= 1 to n_1 do
for i_2:= 1 to n_2 do
:
for i_m:= 1 to n_m do
$\boxed{k := k + 1}$

Ans. ______________

Sum Rule

k:= 0
for i_1 := 1 to n_1 do
k:= k + 1
for i_2 := 1 to n_2 do
k:= k + 1
:
for i_m:= 1 to n_m do
k:= k + 1

Ans. ________________

Ex 14. BASIC's variable name consists of two alphanumeric, begin with a letter. And 5 strings of two characters are reserved words. How many different variable names are there in BASIC?

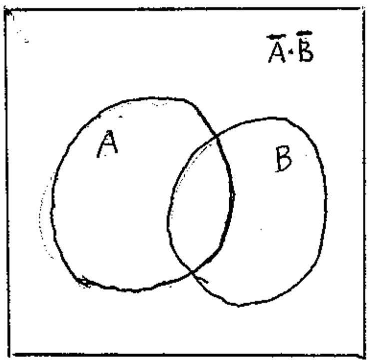
i) one character: 26
ii) two character: 26 (26+10) – 5 = 931

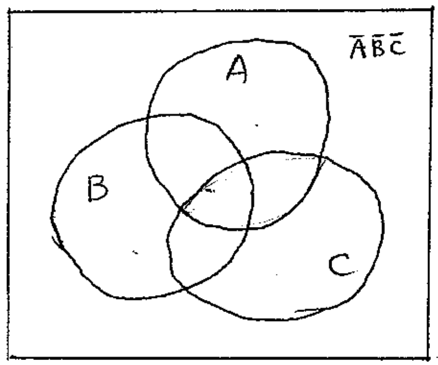
Ans. 26 + 931 = 957 names

Principle of Inclusion-Exclusion

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cdot A_2| + |A_2 A_3| + |A_1 A_3|) + |A_1 A_2 A_3|$$





Ex. Determine the number of positive integers n where $n \le 100$ and n is not divisible by 2, 3, or 5.

divisible by 2: $\lfloor 100/2 \rfloor = 50$
divisible by 3: $\lfloor 100/3 \rfloor = 33$
divisible by 5: $\lfloor 100/5 \rfloor = 20$

divisible by 2·3: $\lfloor 100/6 \rfloor = 16$
divisible by 3·5: $\lfloor 100/15 \rfloor = 6$
divisible by 2·5: $\lfloor 100/10 \rfloor = 10$

divisible by 2·3·5: $\lfloor 100/30 \rfloor = 3$

$$100 - (50+33+20) + (16+6+10) - 3 = 26$$

1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89, 91, 97

THE SIEVE OF ERATOSTHENES

The principle of inclusion–exclusion can be used to find the number of primes not exceeding a specified positive integer. Recall that a composite integer is divisible by a prime not exceeding its square root. So, to find the number of primes not exceeding 100, first note that composite integers not exceeding 100 must have a prime factor not exceeding 10. Because the only primes less than 10 are 2, 3, 5, and 7, the primes not exceeding 100 are these four primes and those positive integers greater than 1 and not exceeding 100 that are divisible by none of 2, 3, 5, or 7. To apply the principle of inclusion–exclusion, let P_1 be the property that an integer is divisible by 2, let P_2 be the property that an integer is divisible by 3, let P_3 be the property that an integer is divisible by 5, and let P_4 be the property that an integer is divisible by 7. Thus, the number of primes not exceeding 100 is

$$4 + N(P_1'P_2'P_3'P_4').$$

Since there are 99 positive integers greater than 1 and not exceeding 100, the principle of inclusion–exclusion shows that

$$\begin{aligned} N(P_1'P_2'P_3'P_4') = 99 &- N(P_1) - N(P_2) - N(P_3) - N(P_4) \\ &+ N(P_1P_2) + N(P_1P_3) + N(P_1P_4) + N(P_2P_3) \\ &+ N(P_2P_4) + N(P_3P_4) \\ &- N(P_1P_2P_3) - N(P_1P_2P_4) - N(P_1P_3P_4) \\ &- N(P_2P_3P_4) + N(P_1P_2P_3P_4). \end{aligned}$$

The number of integers not exceeding 100 (and greater than 1) that are divisible by all the primes in a subset of $\{2, 3, 5, 7\}$ is $\lfloor 100/N \rfloor$, where N is the product of the primes in this subset. (This follows since any two of these primes have no common factor.) Consequently,

$$\begin{aligned} N(P_1'P_2'P_3'P_4') = 99 &- \left\lfloor \frac{100}{2} \right\rfloor - \left\lfloor \frac{100}{3} \right\rfloor - \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{7} \right\rfloor \\ &+ \left\lfloor \frac{100}{2\cdot 3} \right\rfloor + \left\lfloor \frac{100}{2\cdot 5} \right\rfloor + \left\lfloor \frac{100}{2\cdot 7} \right\rfloor + \left\lfloor \frac{100}{3\cdot 5} \right\rfloor \\ &+ \left\lfloor \frac{100}{3\cdot 7} \right\rfloor + \left\lfloor \frac{100}{5\cdot 7} \right\rfloor \\ &- \left\lfloor \frac{100}{2\cdot 3\cdot 5} \right\rfloor - \left\lfloor \frac{100}{2\cdot 3\cdot 7} \right\rfloor - \left\lfloor \frac{100}{2\cdot 5\cdot 7} \right\rfloor - \left\lfloor \frac{100}{3\cdot 5\cdot 7} \right\rfloor \\ &+ \left\lfloor \frac{100}{2\cdot 3\cdot 5\cdot 7} \right\rfloor \\ = 99 &- 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 \\ &- 3 - 2 - 1 - 0 + 0 \\ = 21&. \end{aligned}$$

Hence, there are $4 + 21 = 25$ primes not exceeding 100.

5.2 Pigeonhole Principle

Def. If m pigeons occupy n pigeon holes and m > n, then at least one pigeon hole has two or more pigeons roosting in it.

or

Def. If k+1 objects are placed into k boxes, then there is at least one box containing two or more objects.

Proof. by contradiction.

Ex. How many students must be in class to guarantee that at least two students receive the same letter grades, if there are 8 different letter grades (A, A-, B+, B, B-, C+, C, C-)?

Ans:_______

Thm 2. If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Ex 5. Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Ex 7. How many cards must be selected from a deck of 52 cards to guarantee that at least three cards of the same suits are chosen?

Ans: two cards each x four suits = 8 → at least 9

5.3 Permutations and Combinations

$P(n,r) = n\,(n-1)\,(n-2)\,\ldots\,(n-r+1) = \dfrac{n!}{(n-r)!}$

$C(n,r) = \dfrac{n!}{r!\,(n-r)!}$ $\rightarrow P(n,r) = C(n,r)\cdot r!$

Corollary. $C(n, r) = C(n, n-r)$

Ex 7. How many permutations of the letters ABCDEFGH contain the string ABC?

Ans:______________

Ex 12. How many ways are there to select five players from a 10-member tennis team?

Ans:_______________

Ex. How many ways are there for 8 men and 5 women to stand in a line so that no two women stand next to each other?

m m m m m m m m

w w w w w w w w w

Ans:_______________

5.4 Binomial Coefficients

Motivation.

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Ex 2. What is the expansion of $(x+y)^4$?

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Ex 3. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

<u>Ans.</u>

Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

$$x=1,\ y=1,\ \Rightarrow \quad (1+1)^n = \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} \cdot 1 + \cdots + \binom{n}{n} 1^n$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \qquad \text{q.e.d.}$$

Pascal's Identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

$$\binom{0}{0}$$
$$\binom{1}{0} \ \binom{1}{1}$$
$$\binom{2}{0} \ \binom{2}{1} \ \binom{2}{2}$$
$$\binom{3}{0} \ \binom{3}{1} \ \binom{3}{2} \ \binom{3}{3}$$
$$\binom{4}{0} \ \binom{4}{1} \ \binom{4}{2} \ \binom{4}{3} \ \binom{4}{4}$$
$$\binom{5}{0} \ \binom{5}{1} \ \binom{5}{2} \ \binom{5}{3} \ \binom{5}{4} \ \binom{5}{5}$$
$$\binom{6}{0} \ \binom{6}{1} \ \binom{6}{2} \ \binom{6}{3} \ \binom{6}{4} \ \binom{6}{5} \ \binom{6}{6}$$
$$\binom{7}{0} \ \binom{7}{1} \ \binom{7}{2} \ \binom{7}{3} \ \binom{7}{4} \ \binom{7}{5} \ \binom{7}{6} \ \binom{7}{7}$$
$$\binom{8}{0} \ \binom{8}{1} \ \binom{8}{2} \ \binom{8}{3} \ \binom{8}{4} \ \binom{8}{5} \ \binom{8}{6} \ \binom{8}{7} \ \binom{8}{8}$$
$$\cdots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$
$$\cdots$$

(b)

FIGURE 1 Pascal's Triangle.

Permutation with repetition.

Ex 1. How many strings of length k can be formed from the English alphabet?

$$\overbrace{26\cdot 26\cdot 26 \dots 26}^{k} = 26^{k}$$

Combination with repetition

Ex 2. How many ways are there to select 4 pieces of fruits from a bowl?

A(pple)	O(range)	P(ear)
A	O	P
A	O	P
A	O	P
A	O	P
:	:	:

(Solution)

1-kind		A A A A	O O O O	P P P P
2-kind	(3-1)	A A A O	O O O A	P P P A
		A A A P	O O O P	P P P O
	(2-2)	A A O O	A A P P	P P O O
3-kind		A O P, A	A O P, O	A O P, P

Ans. 15

Theorem. for n kinds, r items, C(n+r-1, r)

Note. C(3+4-1, 4) = C(6,4) = 15

Ex 4. **Bakery cookies: A, B, C, D**
How many different ways can 6 cookies be chosen?

$$C(n+r-1,r) = C(4+6-1, 6) = C(9,6) = C(9,3) = 9 \cdot 8 \cdot 7/3 \cdot 2 = 84$$

Permutation with indistinguishable objects

Ex. **How many different strings can be made by reordering the letters of the word**

S U C C E S S ?

Solution:

$$S_1 \; U \; C_1 \; C_2 \; E \; S_2 \; S_3$$

$$\frac{7!}{3! \cdot 2!} = 420$$

Distribution objects into boxes

Ex. **How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?**

Player 1	Player 2	Player 3	Player 4

$$\binom{52}{5} \cdot \binom{47}{5} \cdot \binom{42}{5} \cdot \binom{37}{5}$$

$$= \frac{52! \; \cancel{47!} \; \cancel{42!} \; \cancel{37!}}{\cancel{47!}\,5! \;\; \cancel{42!}\,5! \;\; \cancel{37!}\,5! \;\; 32!\,5!} = \frac{52!}{5!\,5!\,5!\,5!\,32!}$$