Chapter 6 Discrete Probability

Ex 3. In a lottery, players win a large prize when they match four digits in correct order, four digits selected randomly. What is the probability that a player win the prize?

Ans:

Ex 4. In a lottery, a player wins enormous prize if he/she picks correctly 6 numbers out of 50 numbers (1 to 50). What's the winning probability?

Ans:

Ex 6. Find the probability that a hand of five cards in poker contains four cards of one kind.

Ans:

Thm. $p(\overline{E}) = 1 - p(E)$

Thm. $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$

Ex 10. What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by 2 or 5?

E_1 = divisible by 2 $\quad |E_1| = 50$

E_2 = divisible by 5 $\quad |E_2| = 20$

$|E_1 \cap E_2| = 10$

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

$$= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}$$

Note. $P(E) = \lim_{n\to\infty} \frac{n(E)}{n}$

Axiom of Probability

1. $0 \le p(E) \le 1$ // probability of event E //
2. $P(S) = 1$ // S, sample space //
3. For any sequence of mutually exclusive events, $E_1, E_2, \ldots$, that is, events for which $E_i \cdot E_j = \emptyset$ when $i \neq j$,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{n} P(E_i)$$ // P(E): probability of event E //

Ex. Tossing a coin

$$P(\{H\}) = P(\{T\}) = \frac{1}{2}$$

Ex. Rolling an even number with a die

$$P(\{2,4,6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = 1/6 + 1/6 + 1/6 = 1/2$$

Propositions.

- $P(\overline{E}) = 1 - P(E)$
- If $E \subset F$, then $P(E) \le P(F)$
- $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Conditional Probability

Def. Conditional probability of E given F, is defined as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Ex. A coin is flipped three times. The first outcome was tail.
What is the probability that an odd number of tails appear?

$$\underbrace{TTT}_{odd},\ TTH,\ THT,\ \underbrace{THH}_{odd} \rightarrow 2/4 \quad \text{// conditional probability //}$$

Ex. An urn contains 10 white, 5 yellow, and 10 black marbles.
A marble is chosen at random from the urn, and it is not black.
What is the probability that is yellow?

$$P(Y \mid \bar{B}) = \frac{P(Y \cap \bar{B})}{P(\bar{B})} = \frac{5/25}{15/25} = \frac{1}{3}$$

Independence

Def. The events E and F are independent iff $P(E \cap F) = P(E)P(F)$.

Ex. A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an Ace and F is the event that is a spade, the E and F are independent.

$$P(E \cap F) = 1/52, \quad P(E) = 4/52, \quad P(F) = 13/52$$

Ex. Two coins are flipped.
E = first coin lands head
F = second coin lands tail

E and F are independent.
$P(E \cap F) = (\{H,T\}) = 1/4$
$P(E) = P(\{(H,H), (H,T)\}) = 1/2$
$P(F) = P(\{(H,T), (T,T)\}) = 1/2$

Bernoulli Trials

- Only two possible outcome: success (p), failure (q)
- $p + q = 1$ or $q = 1 - p$

Thm. The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

$$\binom{n}{k} p^k q^{n-k}$$

Ex. What is the probability that exactly 4 heads come up when the coin is flipped 8 times, assuming that the flips are independent?

Ans:__________________

Random Variable

Def. **Real-valued function defined on the sample space of an experiment. That is, a random variable assigns a real value to each possible outcome.**

Ex 10. **Suppose a coin is flipped three times. Let X(t) be the random variable tat equals the number of heads that appear when t is the outcome. Then**

$X(HHH) = 3$
$X(HTH) = X(HHT) = X(THH) = 2$
$X(HTT) = X(THT) = X(TTH) = 1$
$X(TTT) = 0$

Def. **Distribution of a random variable X on a sample space S is the set of pairs (r, P(X=r) for all r ∈ X(S), where P(X=r) is the probability that X takes the value r. A distribution is usually described by specifying P(X=r) for each r ∈ X(S).**

$P\{X = 3\} = 1/8$
$P\{X = 2\} = 3/8$
$P\{X = 1\} = 3/8$
$P\{X = 0\} = 1/8$

Birthday Problem

- **minimum number of people so that at least two of them have the same birthday is greater than 1/2.**

Pn: probability that n people have different birthday

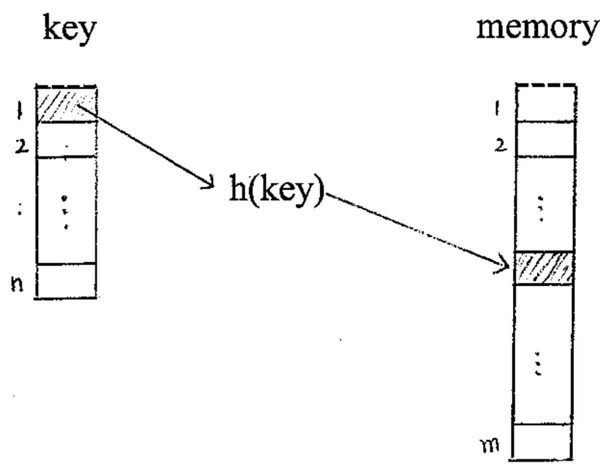
$$1 - P_n = 1 - \frac{366}{366} \cdot \frac{365}{366} \cdot \frac{364}{366} \cdot \frac{363}{366} \cdots \frac{367-n}{366} \geq 0.5$$

Ans. $n = 23$ **(0.506)**

TABLE 9.2
BIRTHDAY PROBLEM RESULTS

N	Theoretical	Simulation
5	.027	.028
10	.117	.110
15	.253	.255
20	.411	.412
22	.476	.462
23	.507	.520
25	.569	.553
30	.706	.692
35	.814	.819
40	.891	.885
45	.941	.936
50	.970	.977
55	.986	.987

Ex. 14 Probability of a collision in hashing function



$n < m$

Case of no collision

	Pr(no collision)
1st key :	$m / m = 1$
2nd key :	$(m-1)/m$
3rd key :	$(m-2)/m$
:	:
jth key :	$(m-(j-1))/m = (m-j+1)/m$

P_n (no collision up to n-th key) $= \frac{m-1}{m} \cdot \frac{m-2}{m} \cdots \frac{m-n+1}{m}$

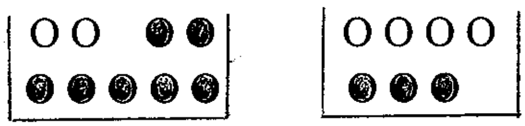
P (a collision) $= 1 - P_n$

$$= 1 - \frac{m-1}{m} \cdot \frac{m-2}{m} \cdots \frac{m-n+1}{m}$$

Bayes' Theorem

$$P(F \mid E) = \frac{P(E|F) \cdot P(F)}{P(E|F)P(F) + P(E|\bar{F}) \cdot P(\bar{F})}$$

Ex.



- Choose one of the two boxes at random, and select one of the balls in that box at random.

Problem. If a black ball is selected, what is the probability that the ball is from the first box? (Is it 7/10?)

Solution.

E: a black ball is selected

$\bar{E}$: a white ball is selected

F: the ball is from the first box

$\bar{F}$: the ball is from the second box

We want to know $P(F \mid E)$.

$P(E \mid F) = 7/9$

$P(E \mid \bar{F}) = 3/7$

$P(F) = P(\bar{F}) = 1/2$

$$P(F \mid E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|\bar{F})P(\bar{F})}$$

$$= \frac{\frac{7}{9} \cdot \frac{1}{2}}{\frac{7}{9} \cdot \frac{1}{2} + \frac{3}{7} \cdot \frac{1}{2}}$$

$$= \frac{\frac{7}{18}}{\frac{38}{63}}$$

$$= \frac{49}{76} \approx \underline{0.645}$$