# Chapter 7 Advanced Counting Techniques

## 7.1 Recurrence Relations (Difference Equations)

**Conditions:**

1. **Termination condition (Initial condition)**
2. **Recurrence relation**
   – express $a_n$ in terms of previous terms
3. **Should reach the termination condition eventually**

**Ex. Compound interst**

$Pn \equiv$ amount of money after n years
$r \equiv$ annual interest rate
$Po \equiv$ principal

$$
\begin{aligned}
P_n &= P_{n-1} + P_{n-1} * r \\
&= (1+r) P_{n-1} \\
&= (1+r) * (1+r) P_{n-2} \\
&\;\vdots \\
&= \underline{(1+r)^n \ P_o}
\end{aligned}
$$

**Ex. Fibonacci Squence : 0, 1, 1, 2, 3, 5, 8, 13, 21, …**

$$
\begin{cases}
F(0) = 0 \\
F(1) = 1 \\
F(n) = F(n-1) + F(n-2), \quad n>1
\end{cases}
$$

$$
\begin{aligned}
F(n) &= F(n-1) + F(n-2) \\
&= F(n-2) + F(n-3) + F(n-3) + F(n-4) \\
&= F(n-2) + 2 \cdot F(n-3) + F(n-4) \\
&= F(n-3) + F(n-4) + 2 \cdot F(n-4) + 2 \cdot F(n-5) + F(n-5) + F(n-6) \\
&= F(n-3) + 3 \cdot F(n-4) + 3 \cdot F(n-5) + F(n-6) \\
&\;\vdots
\end{aligned}
$$

**Note. simple substitution does not help** ($\because$ 2nd order)

# Iterative Method (Substitution Method) for the 1st Order Recurrence Relation

[Ex 1] linear

$$\begin{cases} a_0 = 3, \\ a_n = a_{n-1} + 2 \end{cases} \quad \text{or} \quad F(n) = \begin{cases} 3, & n = 0 \\ F(n-1) + 2, & n \geq 1 \end{cases}$$

Note: $\{\overset{n:\,0}{3}, \overset{1}{5}, \overset{2}{7}, \overset{3}{9}, \ldots\}$

$$\begin{aligned} a_n &= a_{n-1} + 2 \\ &= (a_{n-2} + 2) + 2 = a_{n-2} + (2+2) \\ &= (a_{n-3} + 2) + (2+2) = a_{n-3} + (2+2+2) \\ &= a_{n-4} + (2+2+2+2) \\ &\ \vdots \\ &= a_{n-n} + \overbrace{(2+2+\cdots+2)}^{n \text{ times}} \\ &= a_0 + n \cdot 2 \\ &= 3 + n \cdot 2 \end{aligned}$$

$$a_n = \boxed{2n+3}$$

[Ex.2] non-linear

$$\begin{cases} a_0 = 1 \\ a_n = a_{n-1} + n \end{cases} \quad \text{or} \quad F(n) = \begin{cases} 1, & n = 0 \\ F(n-1) + n & n \geq 1 \end{cases}$$

Note: $\{1, 2, 4, 7, 11, \ldots\}$

$$\begin{aligned} a_n &= a_{n-1} + n \\ &= a_{n-2} + (n-1) + n \\ &= a_{n-3} + (n-2) + (n-1) + n \\ &\ \vdots \\ &= a_{n-n} + (n-(n-1)) + \cdots + (n-2) + (n-1) + n \\ &= a_0 + (1 + 2 + \cdots + (n-2) + (n-1) + n) \\ &= 1 + (n(n+1)/2 \end{aligned}$$

$$\boxed{a_n = \frac{n(n+1)}{2} + 1}$$

[Ex 3] $\begin{cases} a_1 = 3 \\ a_n = 3 \cdot a_{n-1} + 2n \end{cases}$

Note. In this case, the substitution method requires to find the sum of complex series.

$$\begin{aligned}
a_n &= 3 \cdot a_{n-1} + 2n \\
&= 3\,[3 \cdot a_{n-2} + 2(n-1)] + 2n \\
&= 3^2 a_{n-2} + 3 \cdot 2(n-1) + 2n \\
&= 3^2 [3 \cdot a_{n-3} + 2 \cdot (n-2)] + 3 \cdot 2(n-1) + 2n \\
&= 3^3 a_{n-3} + 3^2 \cdot 2(n-2) + 3 \cdot 2(n-1) + 2n \\
&= 3^3 a_{n-3} + 2\,[3^2(n-2) + 3(n-1) + n] \\
&= 3^4 a_{n-4} + 2\,[3^3(n-3) + 3^2(n-2) + 3(n-1) + n] \\
&\vdots \\
&= 3^{n-1} a_{n-(n-1)} + 2\,[3^{n-2}(2) + 3^{n-3}(3) + \cdots + 3^2(n-2) + 3(n-1) + n] \\
&= 3^n + 2\,\underbrace{[3^{n-2}(2) + 3^{n-3}(3) + \cdots + 3^2(n-2) + 3(n-1) + n]}_{?}
\end{aligned}$$

In general, non-linear recurrence relation is hard to solve.

Note: $a_n = \left(\frac{11}{6}\right) 3^n - n - \frac{3}{2}$

# Solving Linear Homogeneous Recurrence Relation with Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \qquad \text{(degree k or k-th order)}$$

**Plug in** $a_n = r^n$

Then $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$

Divide by $r^{n-k}$, move to the left.

$$\underbrace{r^k - c_1 r^{k-1} - c_2 r^{k-2} \dots - c_{k-1} r - c_k = 0}_{k \text{ roots}} \quad : \text{characteristic equation}$$

$a_n$ is linear combination of k roots of the characteristic equation.

*Degree 2 (2nd order) case*:

Theorem 1

Let $C_1$ and $C_2$ be real numbers. Suppose $r^2 - C_1 r - C_2 = 0$ has two distinct roots $r_1$ and $r_2$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

Note. In differential equation case,

$$\frac{d^2F}{dt^2} + \frac{dF}{dt} + F = 0, \quad \text{plug in } F = e^{kt} \text{ and solve}$$

Example. $\begin{cases} a_n = a_{n-1} + 2a_{n-2} \\ a_0 = 2, \ a_1 = 7 \end{cases}$

Characteristic equation: $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \Rightarrow r_1 = 2, r_2 = -1$

$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$

$\left.\begin{array}{l} a_0 = 2 = \alpha_1 + \alpha_2 \\ a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2(-1) \end{array}\right\} \alpha_1 = 3, \alpha_2 = -1$

$\therefore \boxed{a_n = 3 \cdot 2^n - (-1)^n}$

Theorem 2

When the characteristic equation has only one root, $r_0$, then

$$\underline{a_n = \alpha_1 r_0^n + \alpha_2 n \cdot r_0^n}$$

Example.5 $\begin{cases} a_n = 6 \cdot a_{n-1} - 9a_{n-2} \\ a_0 = 1, \ a_1 = 6 \end{cases}$

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0 \Rightarrow r = 3$

$a_n = \alpha_1 3^n + \alpha_2 n \cdot 3^n$

boundary: $\left.\begin{array}{l} a_0 = 1 = \alpha_1 \\ a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3 \end{array}\right\} \alpha_1 = 1, \alpha_2 = 1$

$\therefore \boxed{a_n = 3^n + n \cdot 3^n} = (n+1) \cdot 3^n$

# Ex 4. **Fibonacci Numbers** (p. 463)

0, 1, 1, 2, 3, 5, 8, 13, 21, …

$$\begin{cases} f_0 = 0, \quad f_1 = 1 & \text{:initial conditions} \\ f_n = f_{n-1} + f_{n-2} \quad n \geq 2 & \text{: recurrence} \end{cases}$$

<u>Note</u>. ordinary, linear, constant-coefficient, homogeneous, 2nd-order difference equation

$$f_n - f_{n-1} - f_{n-2} = 0$$

Let $f_n = r^n$

$$r^n - r^{n-1} - r^{n-2} = 0$$

$$r^{n-2}\underline{(r^2 - r - 1)} = 0 \qquad \text{: characteristic equation}$$

$$r = \frac{1 \pm \sqrt{5}}{2}, \quad r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

From initial conditions,

$$\begin{cases} f_0 = A + B = 0 \\ f_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

$$A = \frac{1}{\sqrt{5}} \qquad B = -\frac{1}{\sqrt{5}}$$

$$\therefore \boxed{f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

Note. $\frac{1+\sqrt{5}}{2} \fallingdotseq 1.62$ : golden ratio

Theorem 3.

When the characteristic equation has $k$ roots, $r_1, r_2 \cdots r_k$.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

Ex 6 3rd-order recurrence relation

$$\begin{cases} a_0 = 2, \ a_1 = 5, \ a_2 = 15 \\ a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \end{cases}$$

Characteristic equation:

$$r^3 - 6r^2 + 11r - 6 = 0$$

$$(r-1)(r-2)(r-3) = 0$$

$$r_1 = 1, \ r_2 = 2, \ r_3 = 3$$

$$a_n = A(1)^n + B(2)^n + C\cdot(3)^n$$

$$\begin{cases} 2 = A + B + C & \text{——(1)} \\ 5 = A + 2B + 3C & \text{——(2)} \\ 15 = A + 4B + 9C & \text{——(3)} \end{cases}$$

$(2)-(1)$: $3 = B + 2C$ ——(4)

$(3)-(1)$: $13 = 3B + 3C$ ——(5)

$(5)-3\times(4)$ $\quad 4 = 2C \implies C = 2$

$B = -1$

$A = 1$

$$\boxed{a_n = 1 - 2^n + 2\cdot 3^n}$$

# Nonhomogeneous Recurrence Relation

$$\underbrace{a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + a_k a_{n-k}}_{\text{associated homogeneous recurrence relation}} + F(n)$$

**Thm** General solution = homogeneous solution + particular solution.

**Ex 10.**

$$\begin{cases} a_1 = 3 \\ a_n = 3a_{n-1} + 2n \end{cases} \qquad // \{3, 13, 45, \cdots\} //$$

Homogeneous solution:

$$a_n - 3a_{n-1} = 0$$
$$r^n - 3 \cdot r^{n-1} = 0$$
$$r^{n-1}(r-3) = 0 \Rightarrow r = 3$$

$$\boxed{a_n^{(h)} = A \cdot 3^n}$$

Let particular solution $H_p = cn + d$

$$cn + d = 3(c(n-1) + d) + 2n$$
$$= 3(cn - c + d) + 2n$$
$$= 3cn - 3c + 3d + 2n$$

$$\begin{cases} 2cn + 2n + 2d - 3c = 0 \\ (2c+2)n + (2d - 3c) = 0 \end{cases}$$

$$2c + 2 = 0 \Longrightarrow c = -1$$
$$2d - 3c = 0 \Longrightarrow d = -\frac{3}{2}$$

$$\therefore a_n^{(h)} = -n - \frac{3}{2}$$

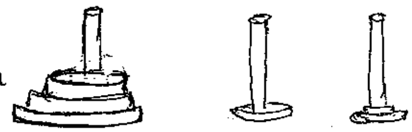
$$\therefore a_n = A \cdot 3^n - n - \frac{3}{2}$$

When $n = 1$, $a_n = 3$.

$$A \cdot 3 - 1 - \frac{3}{2} = 3 \Rightarrow A = \frac{11}{6}$$

$$\therefore \boxed{a_n = \frac{11}{6} \cdot 3^n - n - \frac{3}{2}}$$

## Ex 5. Tower of Hanoi (P.452)




$n=1$ : 1 move
$n=2$ : 3 moves
$n=3$ : 7 moves
$n=4$ : 15 moves
⋮ ⋮

```
proc HANOI (n, A, B, C)
    if n=1 then move the disk from A to C
           else { call HANOI (n-1, A, C, B)
                  move disk n from A to C
                  call HANOI (n-1, B, A, C)
```

Recurence: $H(n) = \begin{cases} 1 & : n=1 \\ 2H(n-1)+1 & : n>1 \end{cases}$

[Method 1] substitute

$$\begin{aligned} H(n) &= 2 \cdot H(n-1)+1 \\ &= 2[2H(n-2)+1]+1 \\ &= 2^2 H(n-2) + 2+1 \\ &= 2^2 [2H(n-3)+1]+2+1 \\ &= 2^3 H(n-3) + 2^2 + 2+1 \\ &\vdots \\ &= 2^{n-1} \cdot H(1) + 2^{n-2} + 2^{n-3} + \cdots + 4 + 2 + 1 \\ &= \boxed{2^n - 1} \end{aligned}$$

[Method 2] $H(n) - 2H(n-1) - 1 = 0$

Homogeneous: $H_h(n) - 2H_h(n-1) = 0$

$$r^n - 2r^{n-1} = 0$$
$$r^{n-1}(r-2) = 0 \Rightarrow r=2$$
$$\therefore \boxed{H_h(n) = A \cdot 2^n}$$

particular: $H_p(n) - 2H_p(n-1) = 1$

$$\boxed{H_p(n) = -1}$$

General: $H(n) = A \cdot 2^n - 1$

$H(1) = 1 \Rightarrow A = 1$

$$\therefore \boxed{H(n) = 2^n - 1}$$

Note: If one move takes 1 second, for $n=64$,

$2^{64} - 1 \fallingdotseq 2 \times 10^{19}$ sec

$\fallingdotseq$ 500 billion years!

# Ex 8. Catalan Number $C_n$ (P. 456)

[1] **Chained Matrix Product**

$$M_1 * M_2 * M_3 * \ldots * M_n$$

**How many different ways can we perform these multiplications?**

**n = 3:**

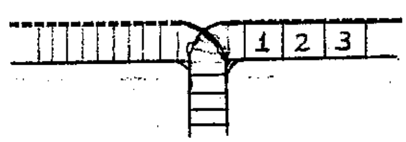
$(M_1 * M_2) * M_3$
$M_1 * (M_2 * M_3)$

**n = 4:**

$((M_1 * M_2) * M_3) * M_4$
$(M_1 * (M_2 * M_3)) * M_4$
$M_1 * ((M_2 * M_3) * M_4)$
$(M_1 * (M_2 * (M_3 * M_4)))$
$((M_1 * M_2) * (M_3 * M_4))$

Railroad switching network

[2] **Given a stack of depth n and a sequence of elements 1,2,3,...n. Find the number of different permutations obtained by a sequence of n pushes and n pops.**




| | |
|---|---|
| push push push pop pop pop | 3 2 1 |
| push pop push push pop pop | 1 3 2 |
| push push pop push pop pop | 2 3 1 |
| push push pop pop push pop | 2 1 3 |
| push pop push pop push pop | 1 2 3 |

**Note. (3 1 2) is not possible**

[3] **Given n pairs of parenthesis, find the number of distinct well-formed expressions.**

$S \to SS \mid (S) \mid \lambda$

**Ex. n = 3**

( ( ) ( ) ) : **push push pop push pop pop**
( ) ( ) ( )
( ) ( ( ) )
( ( ( ) ) )
( ( ) ) ( )
$\leftarrow 2n \rightarrow$

**Condition: at any position i, $1 \le i \le 2n$, #L.P. $\ge$ #R.P.**

**# of valid distinct expressions = # of all expressions − # of invalid express**

$$= \binom{2n}{n} - \binom{2n}{n-1} = \frac{2n!}{n!\,n!}\left[1 - \frac{n}{(n+1)}\right]$$

$$= \frac{1}{n+1}\binom{2n}{n} \approx \frac{4^n}{n^{3/2}}$$

EX.

invalid

( ( ) ( ) ) ) ( ) ( (

) ) ( ) ( ( ( ( ) ( (

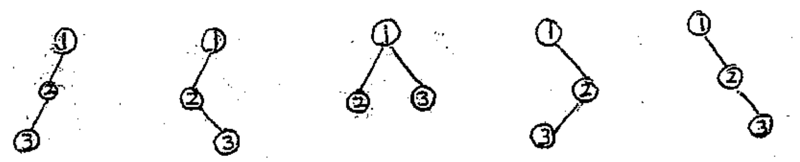
(n+1) l.P.
(n−1) r.P. $\Rightarrow$ $\binom{2n}{n-1}$ is total # of invalid expression

[4] **Let $T_n$ be # of distinct binary trees with n nodes.**

$T_0 = 1$

$T_1 = 1$ 

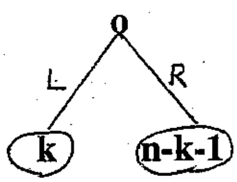
$T_2 = 2$ 

$T_3 = 5$ 

Inorder traversal: (3, 2, 1) (2, 3, 1) (2, 1, 3) (1, 3, 2) (1, 2, 3)

Note

**root**




$\equiv$ $(\underbrace{(())}_{k \text{ pair}}\ \underbrace{((()))}_{n-k-1 \text{ pair}})$

$$T_n = \sum_{k=0}^{n-1} T_k \cdot T_{n-k-1} = \frac{1}{n+1}\binom{2n}{n}$$

**Ex.**

$$\begin{aligned} T_3 = \sum_{k=0}^{2} T_k \cdot T_{2-k} &= T_0 \cdot T_2 + T_1 \cdot T_1 + T_2 \cdot T_0 \\ &= 1\cdot 2 \quad 1\cdot 1 \quad 2\cdot 1 \\ &= 5 \end{aligned}$$

**Catalan sequence: 1, 1, 2, 5, ...**

## 7.3 Divide-and-Conquer Algorithm and Recurrence Relation

**Recurrence relation:**

$$f(n) = a \cdot f(n/b) + g(n)$$

**Ex. 1 Binary search**

$$T(n) = T(n/2) + 2$$

**Ex. 2 Merge Sort**

$$T(n) = \begin{cases} 1 & n = 2 \\ 2\,T(n/2) + (n-1) & n > 2 \end{cases}$$

**Ex 3. FFT (Fast Fourier Transformation)**

$$T(n) = \begin{cases} 0 & n = 1 \\ 2\,T(n/2) + 3/2\,n & n > 2 \end{cases}$$

[EX 4] Multiplying two large integers   cryptography

$$\overset{a}{35}\,|\,\overset{b}{12} \times \overset{c}{24}\,|\,\overset{d}{32}$$

$$= \underbrace{(35\times 24)}_{M_1}\cdot 10^4 + \underbrace{(12\times 32)}_{M_2} + \underbrace{(35\times 32 + 24\times 12)}_{\underbrace{(35+12)\times(24+32)}_{M_3} - \underbrace{(35\times 24)}_{M_1} - \underbrace{(12\times 32)}_{M_2}}\cdot 10^2$$

$$= \boxed{M_1\times 10^4 + M_2 + (M_3 - M_1 - M_2)\times 10^2}$$

one 4-digit multiplication ⇒ Three 2-digit multiplication

$$6\begin{cases} \text{+ Four } \text{''} & + \\ \text{+ Two } \text{''} & - \end{cases}\Big\}\ t_1$$

+ Two shifts } $t_2$

$$\overset{\leftarrow n \rightarrow}{a\,|\,b} * \overset{\leftarrow n \rightarrow}{c\,|\,d}$$

$$= a*c\cdot 10^n + [(a+b)*(c+d) - a*c - b*d]\cdot 10^{\frac{n}{2}} + b*d.$$

additional work : $6\,t_1\frac{n}{2} + 2\,t_2\cdot n = (3t_1+2t_2)n = kn$

$$T(n) = \begin{cases} 1 & n=1 \\ 3T(\frac{n}{2}) + kn, & n\geq 2 \end{cases}$$

Let $n = 2^p$, $(p = \log_2 n)$

$$T(n) = 3T(\tfrac{n}{2}) + kn$$
$$= 3[3T(\tfrac{n}{2^2}) + k(\tfrac{n}{2})] + kn$$
$$= 3^2T(\tfrac{n}{2^2}) + \tfrac{3}{2}kn + kn$$
$$\cancel{= 3^2[3T(\tfrac{n}{2^3}) + k\cdot\tfrac{n}{2^2}] + \tfrac{3}{2}kn + kn}$$
$$= 3^3T(\tfrac{n}{2^3}) + (\tfrac{3}{2})^2kn + (\tfrac{3}{2})kn + kn$$
$$\vdots$$
$$= 3^pT(1) + k\cdot n\boxed{\sum_{i=0}^{p-1}(\tfrac{3}{2})^i} \quad \text{-----} \quad \frac{(\frac{3}{2})^p - 1}{(\frac{3}{2}) - 1} = 2\cdot[(\tfrac{3}{2})^p - 1] = 2(\tfrac{3}{2})^p - 2$$

$$= 3^p + kn[2\cdot(\tfrac{3}{2})^p] - 2kn$$
$$\cancel{= 3^p + 2k\cdot 2^p(\tfrac{3}{2})^p - 2kn}$$
$$= 3^p + 2k\,3^p - 2kn$$
$$= \boxed{3^p}(1+2k) - 2kn \qquad \boxed{3^{\log_2 n} = n^{\log_2 3}} \ \because \log_2 n \log_2 3 = \log_2 3\cdot\log_2 n$$
$$= (1+2k)\cdot n^{\log_2 3} - 2kn$$
$$= O(n^{1.59}) \qquad \text{Karatsuba and Offman (1962)}$$

[EX 5] Matrix Multiplication

$$\overset{n\times n}{A} * \overset{n\times n}{B} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] * \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ \hline A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{array}\right]$$

$$T(n) = \begin{cases} 8\cdot T(\frac{n}{2}) + n^2 & , n>1 \\ 1 & , n=1 \end{cases}$$

Let $k=\log n$

$$\begin{aligned} T(n) &= 8\cdot T(\tfrac{n}{2}) + n^2 \\ &= 8\,[8\,T(\tfrac{n}{2^2}) + (\tfrac{n}{2})^2] + n^2 \\ &= 8^2\,T(\tfrac{n}{2^2}) + 8\cdot(\tfrac{n}{2})^2 + n^2 \\ &= 8^2\,[8\cdot T(\tfrac{n}{2^3}) + (\tfrac{n}{2^2})] + 8\,(\tfrac{n}{2})^2 + n^2 \\ &= 8^3\,T(\tfrac{n}{2^3}) + 8^2(\tfrac{n}{2^2})^2 + 8\,(\tfrac{n}{2})^2 + n^2 \\ &\;\vdots \\ &= 8^k\,T(\tfrac{n}{2^k}) + \underbrace{\sum_{i=0}^{k-1} 8^i \left(\tfrac{n}{2^i}\right)^2}_{=\, n^2\,[1+2+4+8+\cdots+\frac{n}{2}] \,=\, n^2(n-1)} \\ &= n^3 + n^2(n-1) \\ &= O(n^3) \end{aligned}$$

Strassen discovered a clever method of multiplying two 2x2 matrices using 7 multiplications and 18 addition/subtractions.

$$\therefore\ T(n) = 7\cdot T(\tfrac{n}{2}) + 18\,(\tfrac{n}{2})^2$$

$$= O(n^{\log_2 7})$$

$$= O(n^{2.81})$$

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$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$\left.\begin{aligned} m_1 &= (a_{12}-a_{22})(b_{21}+b_{22}) \\ m_2 &= (a_{11}+a_{22})(b_{11}+b_{22}) \\ m_3 &= (a_{11}-a_{21})(b_{11}+b_{12}) \\ m_4 &= (a_{11}+a_{12})\cdot b_{22} \\ m_5 &= a_{11}\,(b_{12}-b_{22}) \\ m_6 &= a_{22}\,(b_{21}-b_{11}) \\ m_7 &= (a_{21}+a_{22})\cdot b_{11} \end{aligned}\right\}\qquad \begin{aligned} C_{11} &= m_1+m_2-m_4+m_6 \\ C_{12} &= m_4+m_5 \\ C_{21} &= m_6+m_7 \\ C_{22} &= m_2-m_3+m_5-m_7 \end{aligned}$$

Hopcroft and Kerr (1971)

- showed 7 multiplications are necessary.

Winograd (1973)

- 7 multiplications, 15 additions/subtractions

Glover

- 36 different ways for A x B using 7 multiplications

Victor Pan: O( n ^ 2.795)

Coppersmith and Winograd (1986): O(n ^ 2.376)

Note.

2 x 2: $n^{\log_{\underset{\uparrow}{2}} \underset{\nwarrow}{7}}$

splitting multiplication

4 x 4: $n^{\log_4 q} < n^{\log_2 7}$

if q < 49 then better than Strassen's

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Ex. FFT

$$T(n) = \begin{cases} 0 & n = 1 \\ 2 \cdot T\left(\frac{n}{2}\right) + \frac{3}{2} n & n \geq 2 \end{cases}$$

Ex. Mergesort

$$T(n) = \begin{cases} 1 & n = 2 \\ 2 \cdot T\left(\frac{n}{2}\right) + (n-1), & n > 2 \end{cases}$$