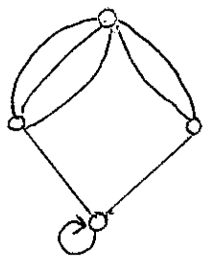
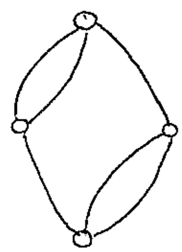
9. Graph G (V, E)

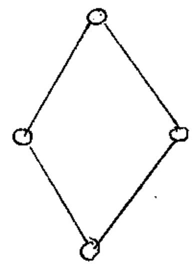
V – vertices(nodes) |V| = n
E – edges |E| = m



Pseudo graph

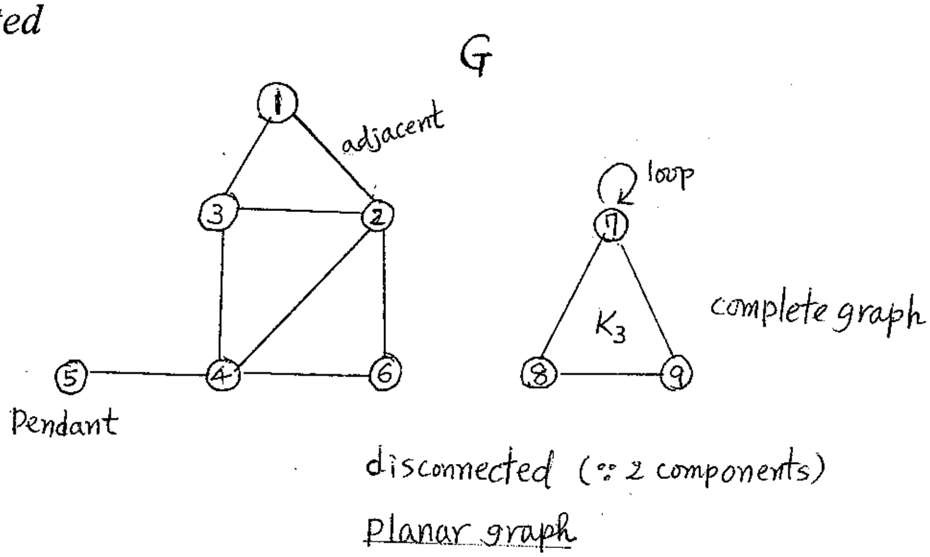


Multigraph



Simple graph

Directed
Undirected



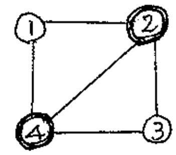
Distance: dist (②,⑤) = 2 length of shortest path
Diameter(G) = maximum distance between any two vertices

Degree: d(1) = 2, d(2) = 4, d(3) = 3

Theorem 1. $\sum_{v \in V} d(v) = 2m$ (* even number *)

Proof. each edge – 2 vertices
So each edge is counted twice in $\sum_{v \in V} d(v)$.

Theorem 2. Number of vertices of odd degree is even.



Proof

① ⟶ ②
predecessor successor

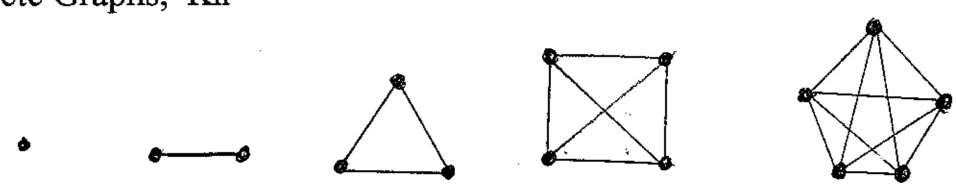
① —— ②
adjacent

path: ①—②—④—③ (simple path)

circuit: ①—②—③—④—① (simple circuit = cycle)

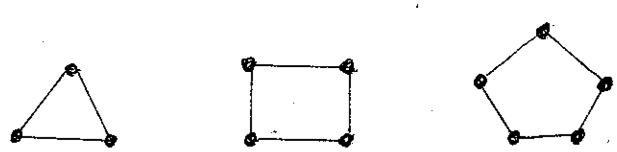
Hamiltonian cycle ≡ A cycle that contains every vertex in the graph

Complete Graphs, K_n



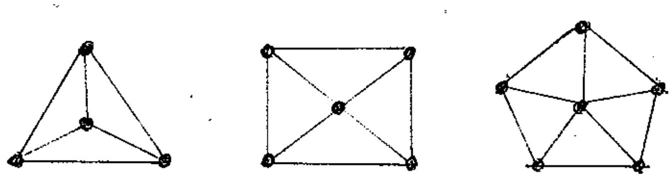
K_1 K_2 K_3 K_4 K_5

Cycles, C_n, n> 3



C_3 C_4 C_5

Wheels



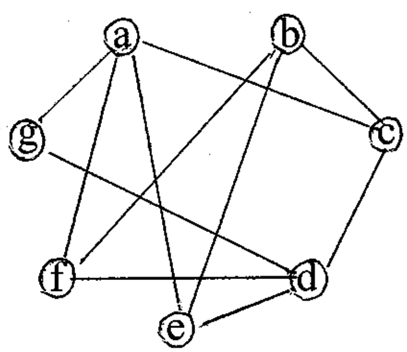
W W W

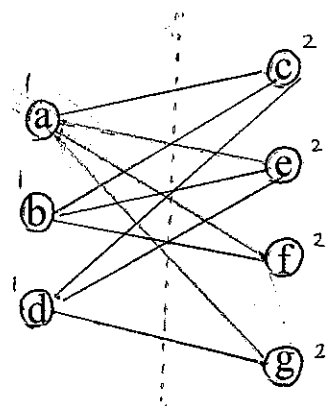
Bipartite Graph

- Vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2

Note. A graph is bipartite iff it is possible to color the vertices of the graph with 2 colors so that no two adjacent vertices have the same color.

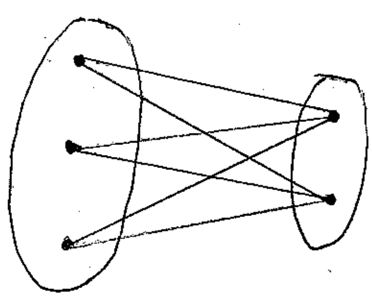
Ex.





Bipartite

Complete Bipartite

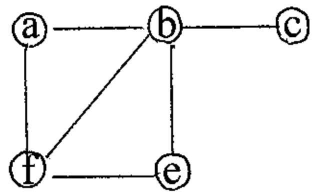


$K_{3,2}$

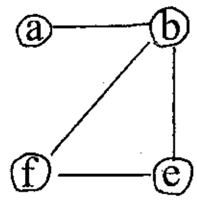
Subgraph

Def. A subgraph of a graph G(V,E) is a graph G' = (V', E') where $V' \subseteq V$ and $E' \subseteq E$.

Ex.



G

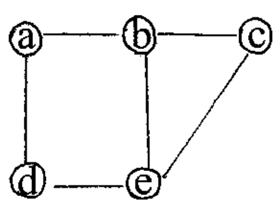


G'

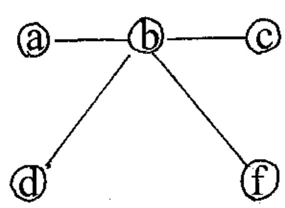
Union

Def. The union of two simple graph $G1 = (V_1, E_1)$ and $G2 = (V_2, E_2)$ is the simple graph with $V_1 \cup V_2$ and $E_1 \cup E_2$.

Ex.



∪



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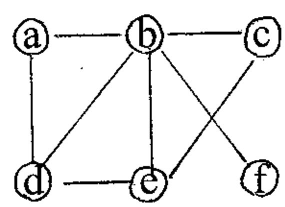
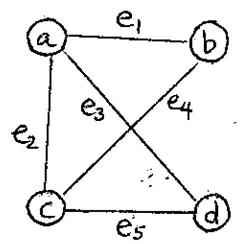


9.3 Graph Representation



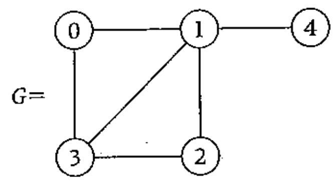
(1) Adjacency Matrix

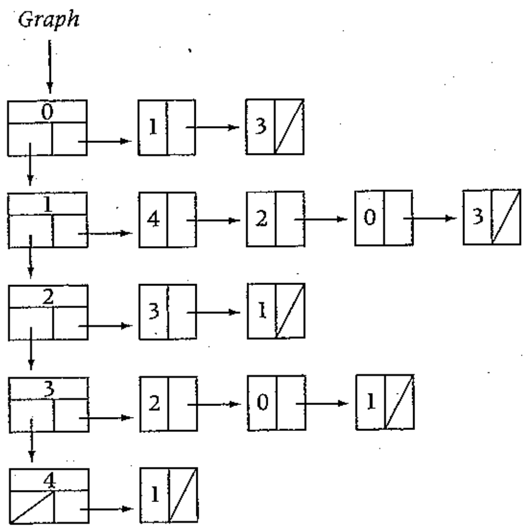
	a	b	c	d
a	0	1	1	1
b	1	0	1	0
c	1	1	0	1
d	1	0	1	0

(2) Incidence Matrix

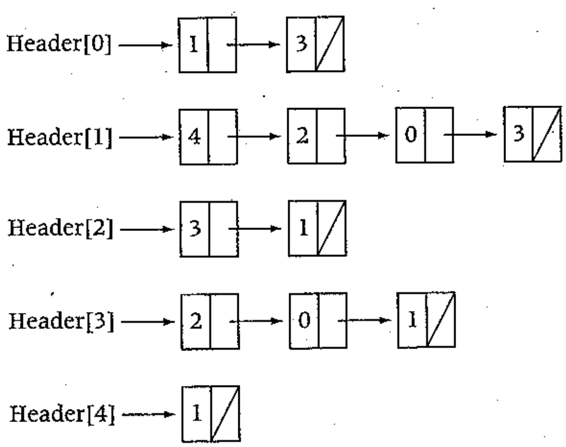
	e_1	e_2	e_3	e_4	e_5
a	1	1	1	0	0
b	1	0	0	1	0
c	0	1	0	1	1
d	0	0	1	0	1

(3) Adjacency List





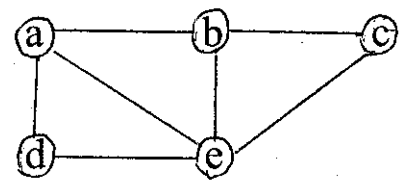
(a) Linked list of header nodes



(b) Array of header nodes

9.4 Connectivity

Path

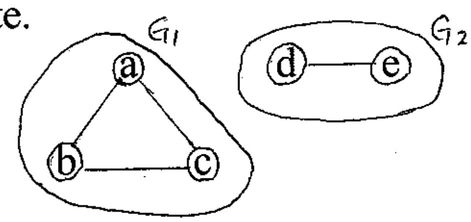


path: (a)—(d)—(e)—(c) simple path of length 3

cycle (circuit): (a)—(d)—(e)—(b)—(a) cycle of length 4

Def. G is connected if there is a path between every pair of vertices.

Note.

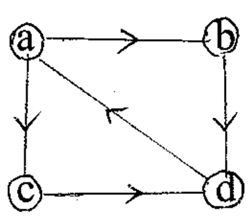


G consists of two connected components G_1 and G_2.

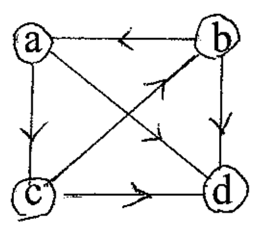
Connectedness in Directed Graph

Def. A digraph is "strongly connected" if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

Def. A digraph is "weakly connected" if there is a path between every two vertices in the underlying undirected graph.



S

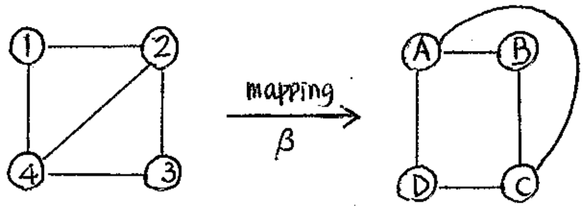


W

Graph Isomorphism Problem

Invariants - same number of vertices, same number of edges, ..

Examples

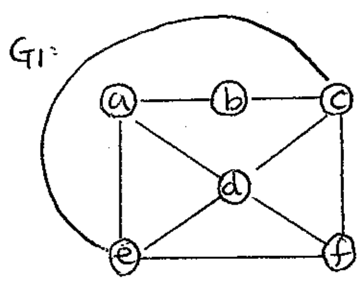


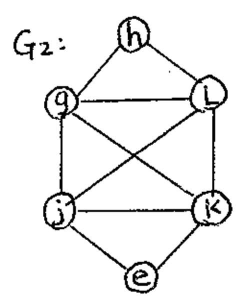
G G'

β: (2, A)
(4, C)
(1, B)
(3, D)
isomorphism

Yes

No





Vertex degree
(descending order)

G_1	G_2
4	4
4	4
4	4
3	4
3	2
2	2

G_1: $|V| = 6$
$|E| = 10$

G_2: $|V| = 6$
$|E| = 10$

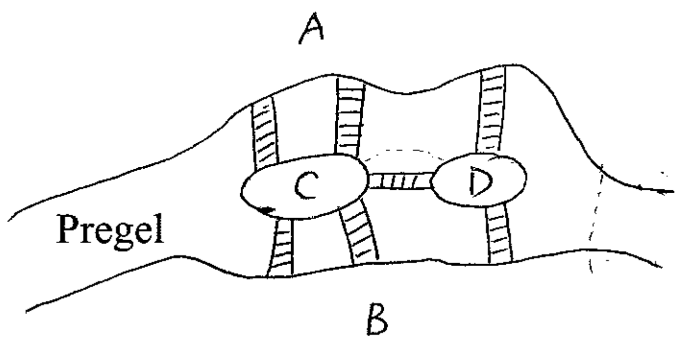
No

No polynomial-time algorithm

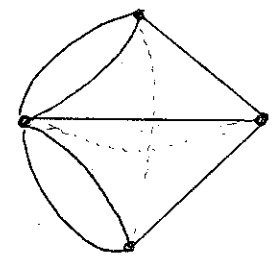
9.5 Euler Path and Hamiltonian Path

Königsberg Bridge Problem (1736)

(East Prussia)



dual



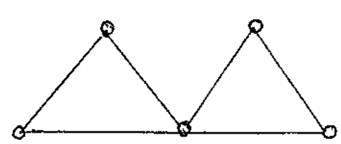
Can you draw the above figure w/o lifting the pen from the paper and w/o retracing a line?

Leonhard Euler (1707-1783)

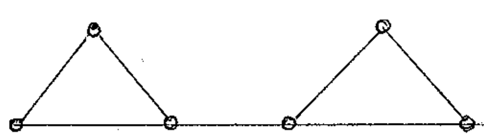
Euler circuit (tour) ≡ a simple circuit containing every edge of G
Euler path ≡ a simple path containing every edge of G

Chinese Postman Problem

Euler tour



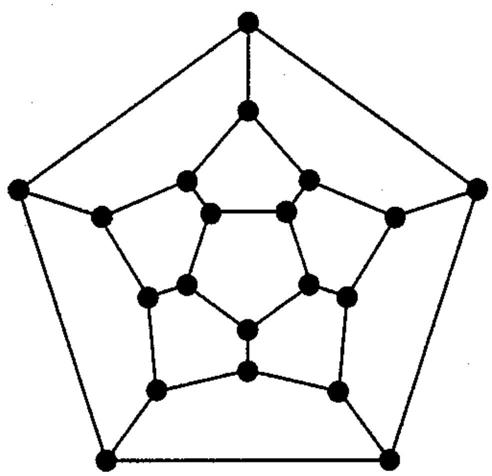
No Euler tour

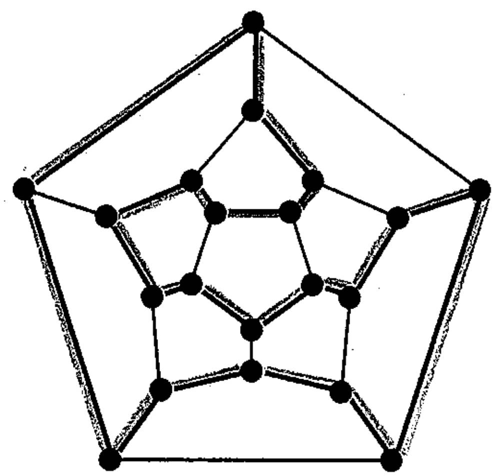


Hamiltonian Path and Circuit

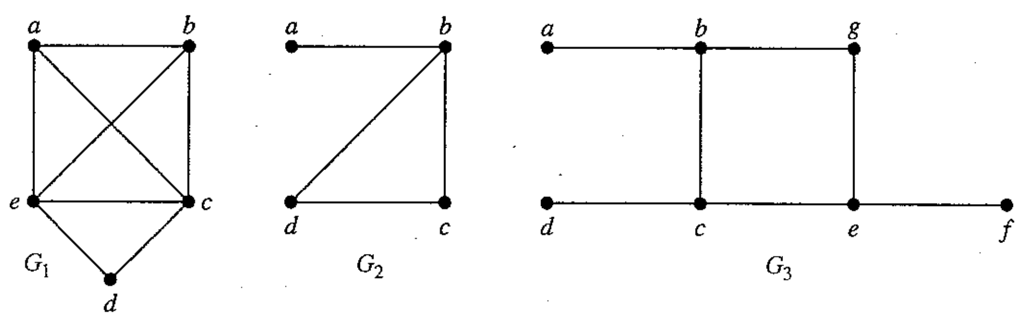
Hamilton's Dodecahedron Puzzle (1857)

- NP-Complete





Ex. 5



9.6 Shortest Path Problem

Shortest Path Problems

- One-to-all SP (Single source SP)
- All-to-all SP

Dijkstra's Algorithm (1959)

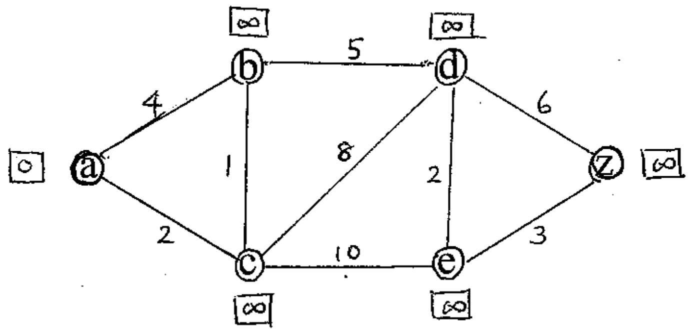
- one-to-all shortest path algorithm
- label-setting algorithm: no negative-weight edges
- $O(n^2)$

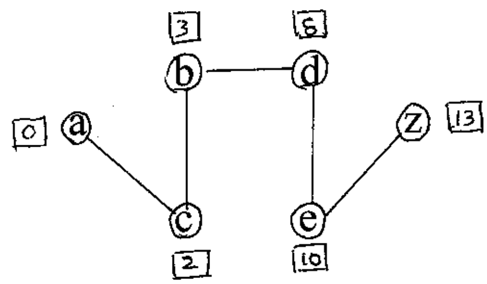
Procedure Dijkstra (G, a)

```
// Initialization Step
    for all vertices v
        Label(v) := ∞
        Pred(v)  := -1
    endfor
    Label(a) := 0   // a is the source node
    S := ∅          // set of nodes whose labels have been set.

// Iteration Step
    while z ∉ S
        u := a vertex not in S with minimal Label
        S := S ∪ {u}
        for all vertices v not in S
          if (Label(u) + Wt(u,v)) < Label(v)     // shorter path is found
            then begin
                    Label(v) := Label(u) + Wt(u,v)
                    Pred(v) := u
                 end
    endwhile
```

Ex. 2



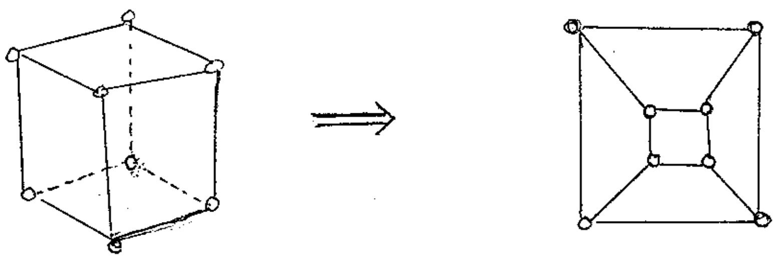


shortest path tree

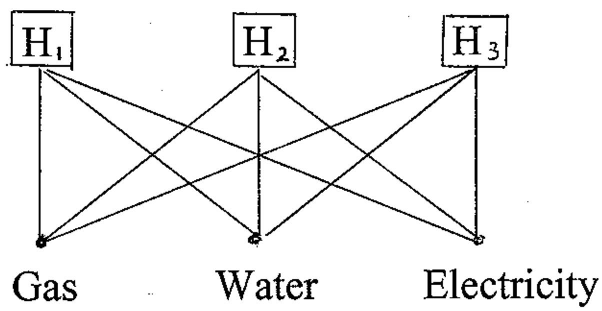
S	a	b	c	d	e	z
-	0	∞	∞	∞	∞	∞
a	0	4	2	∞	∞	∞
c	0	3	2	10	12	∞
b	0	3	2	8	12	∞
d	0	3	2	8	10	14
e	0	3	2	8	10	13
z	0	3	2	8	10	13

9.7 Planar Graph

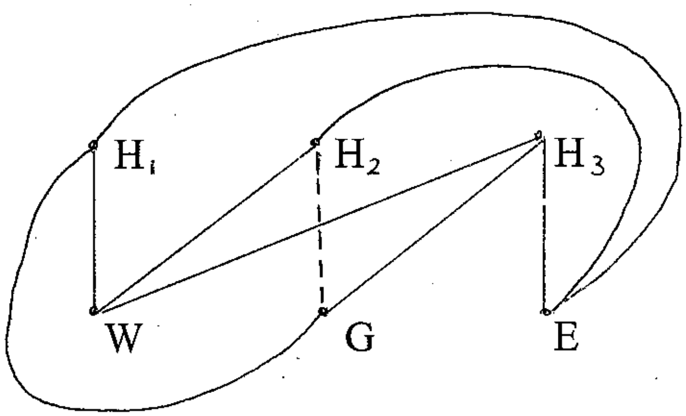
Planar



Three Houses and Three Utilities Problem



Is it possible to join three houses and utilities so that none of
The connections cross? -- Planar graph



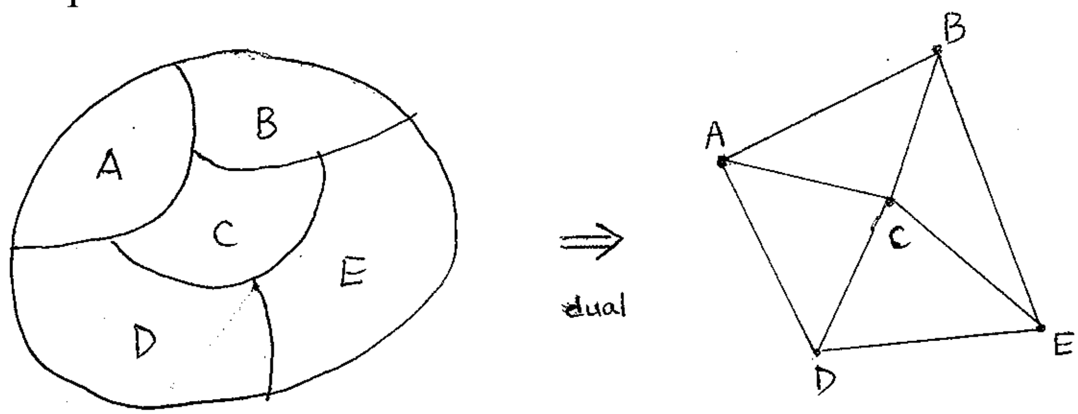
$K_{3,3}$

Kuratowski's Theorem

G is non-planar iff it contains a subgraph that is (isomorphic to) a subdivision of K_5 or $K_{3,3}$

9.8 Graph Coloring

Dual Graph



Def. Coloring of a graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Def. The <u>chromatic number</u> of a graph is the minimum number of colors needed for coloring of this graph.

Four Color Theorem

The chromatic number of a planar graph is 4.
(Four colors are enough to color countries on a world map.)

1856 Conjectured by Guthrie
1976 Proved by Appel and Haken (U of Illinois) - computer