Eigenvalues and Eigenvectors

Linear System: $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Homogeneous Linear System: $Ax = 0$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

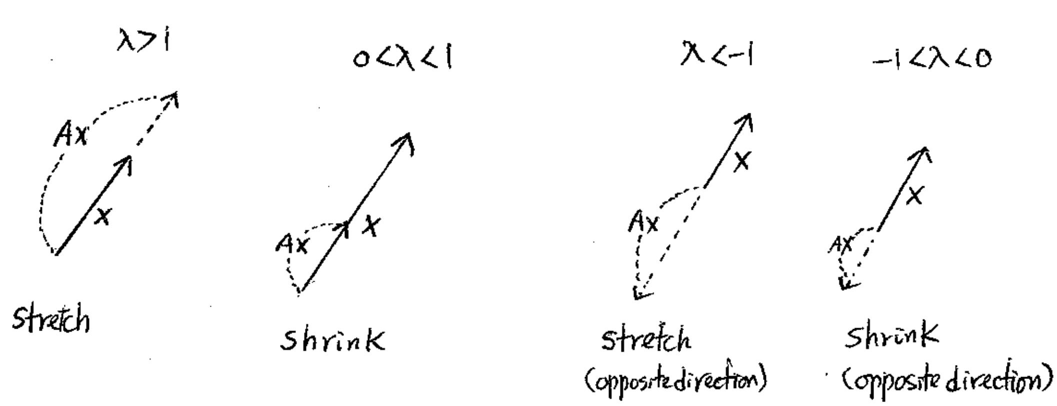
Eigenvalue: $A - \lambda \cdot I = 0$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

If we expand $\det(A - \lambda I)$, we obtain polynomial in λ, $\underline{P(\lambda)}$: characteristic polynomial.
The eigenvalues are the zeros of $P(\lambda)$.
If $(A - \lambda I) \cdot X = 0$, X is called eigenvector.

Note. $Ax = \lambda x$. (A takes the vector X into a scalar multiple of itself.)

<u>case</u>



Ex.1

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$P(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & -1 \\ -1 & 1 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)^2-(-1)\right] + 2(1-\lambda)$$
$$= (1-\lambda)(\lambda^2-2\lambda+2) + 2(1-\lambda)$$
$$= (1-\lambda)(\lambda^2-2\lambda+4)$$

$\therefore \lambda_1 = 1,\ \lambda_2 = 1+\sqrt{3}i,\ \lambda_3 = 1-\sqrt{3}i$

Eigenvector for λ_1,

$$\begin{bmatrix} 1-1 & 0 & 2 \\ 0 & 1-1 & -1 \\ -1 & 1 & 1-1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_3 = 0,\ -2x_3 = 0,\ -x_1 + x_2 = 0$

$x_3 = 0,\ x_2 = x_1$ (x_1 is arbitrary, let $x_1 = 1$)

Eigenvector for $\lambda = 1$, $\underline{(1,1,0)^t}$

Eigenvector for λ_2,

$$\begin{bmatrix} 1-(1+\sqrt{3}i) & 0 & -2 \\ 0 & 1-(1+\sqrt{3}i) & -1 \\ -1 & 1 & 1-(1+\sqrt{3}i) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2 = \left[-\frac{2\sqrt{3}}{3}i,\ \frac{\sqrt{3}}{3}i,\ 1\right]^t$$

Eigenvector for λ_3

$$X_3 = \left[\frac{2\sqrt{3}}{3}i,\ -\frac{\sqrt{3}}{3}i,\ 1\right]^t$$

Def. The spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = \max |\lambda| \quad \text{where } \lambda \text{ is an eigenvalue of } A.$$

For the matrix A in Ex 1,

$$\rho(A) = \max\{1, |1+\sqrt{3}i|, |1-\sqrt{3}i|\} = \max\{1,2,2\} = 2$$

Note. for $\lambda = \alpha + \beta i$, $|\lambda| = \sqrt{(\alpha^2+\beta^2)}$

The spectral radius is closely related to the norm of a matrix.

Thm If A is an $n\times n$ matrix, then

(i) $[\rho(A^TA)]^{\frac{1}{2}} = \|A\|_2$

(ii) $\rho(A) \le \|A\|$ for any natural norm $\|\cdot\|$.

Note. $\rho(A)$ is the greatest lower bound for the natural norm on A.

Ex 2. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$

$\|A\|_\infty = 4$
$\|A\|_1 = 4$
$\|A\|_F = \sqrt{14} \doteqdot 3.74$
$\rho(A) = 2$

$$A^TA = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & 4 \\ -1 & 4 & 5 \end{bmatrix}$$

To calculate $\rho(A^TA)$, we need the eigenvalues of A^TA

$$0 = \det(A^TA - \lambda I) = \det\begin{bmatrix} 3-\lambda & 2 & -1 \\ 2 & 6-\lambda & 4 \\ -1 & 4 & 5-\lambda \end{bmatrix}$$

$$= -\lambda^3 + 14\lambda^2 - 42\lambda$$
$$= -\lambda(\lambda^2 - 14\lambda + 42)$$

$\therefore \lambda_1 = 0,\ \lambda_2 = 7+\sqrt{7},\ \lambda_3 = 7-\sqrt{7}$

$$\|A\|_2 = \sqrt{\rho(A^TA)} = \sqrt{\max\{0, 7+\sqrt{7}, 7-\sqrt{7}\}} \simeq 3.106$$

Def. $N \times N$ matrix A is convergent if

$$\lim_{k\to\infty} (A^k)_{ij} = 0, \quad i=1,\cdots n, j=1,\cdots n.$$

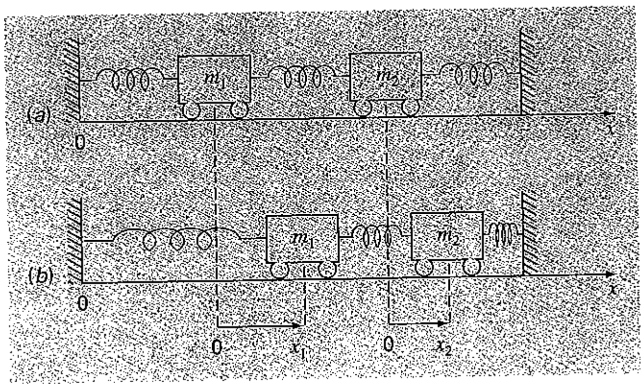
Ex $A = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

$$A^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad A^3 = \begin{bmatrix} \frac{1}{8} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix} \quad A^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ \frac{1}{8} & \frac{1}{16} \end{bmatrix} \cdots A^k = \begin{bmatrix} (\frac{1}{2})^k & 0 \\ \frac{k}{2^{k+1}} & (\frac{1}{2})^k \end{bmatrix}$$

Since $\lim_{k\to\infty} (\frac{1}{2})^k = 0$, and $\lim_{k\to\infty} \frac{k}{2^{k+1}} = 0$, A is convergent matirix.

Note that $\rho(A) = \frac{1}{2}$.

Example. mass-spring system



$$\begin{cases} m_1 \cdot \dfrac{d^2 x_1}{dt^2} = -k x_1 + k(x_2 - x_1) \Rightarrow \\ m_2 \cdot \dfrac{d^2 x_2}{dt^2} = -k x_2 + k(x_1 - x_2) \Rightarrow \end{cases} \boxed{\begin{array}{l} m_1 \dfrac{d^2 x_1}{dt^2} - k(-2x_1 + x_2) = 0 \\ m_2 \dfrac{d^2 x_2}{dt^2} - k(x_1 - 2x_2) = 0 \end{array}} \quad \text{---- (1)}$$

From vibration theory,

$$x_i = X_i \cdot \sin(\omega t) \quad \text{------------------ (2)}$$

where $X_i \equiv$ amplitude of the vibration of mass i

$\omega \equiv$ frequency $(= 2\pi / T_p)$

$$x_i'' = -X_i\, \omega^2 \sin(\omega t) \quad \text{------------------ (3)}$$

Put (2),(3) into (1),

$$\begin{cases} -m_1 X_1 \omega^2 \sin(\omega t) - k(-2X_1 \sin(\omega t) + X_2 \sin(\omega t)) = 0 \\ -m_2 X_2 \omega^2 \sin(\omega t) - k(X_1 \sin(\omega t) - 2 \cdot X_2 \cdot \sin(\omega t)) = 0 \end{cases} \quad \text{---- (4)}$$

Divide (4) by $m_1 \cdot \sin(\omega t)$ and $m_2 \sin(\omega t)$

$$\begin{cases} \left(\dfrac{2k}{m_1} - \omega^2\right) X_1 - \dfrac{k}{m_1} X_2 = 0 \\ -\dfrac{k}{m_1} X_1 + \left(\dfrac{2k}{m_2} - \omega^2\right) X_2 = 0 \end{cases} \quad \text{------------ (5)}$$

This is an eigenvalue problem where $\lambda = \omega^2$.

[1] Polynomial Method

Ex. A.1 Let $m_1 = m_2 = 40$ (kg), and $K = 200$ N/m.

$$\begin{bmatrix} (10-\omega^2) & -5 \\ -5 & (10-\omega^2) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

characteristic polynomial:

$$(10-\omega^2)^2 - (-5)^2 = 0$$

$$100 - 20\omega^2 + (\omega^2)^2 - 25 = 0$$

$$(\omega^2)^2 - 20\omega^2 + 75 = 0.$$

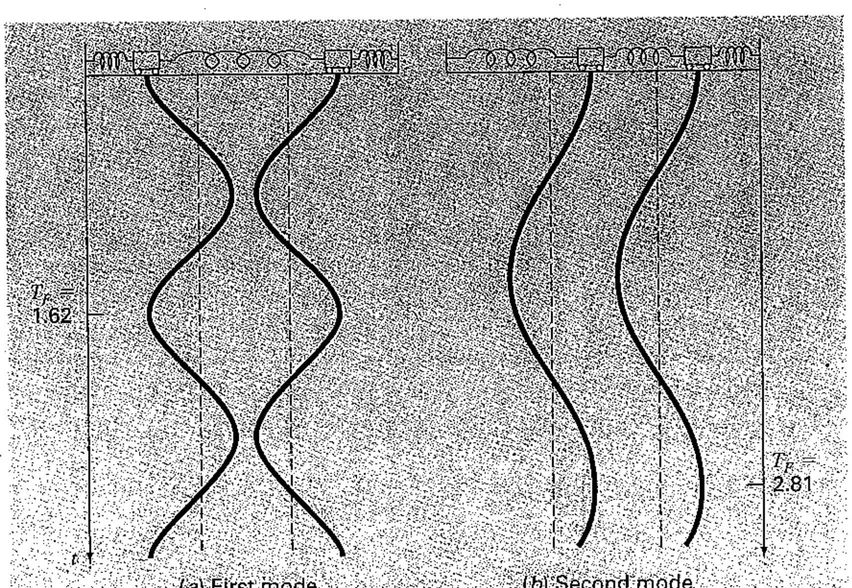
$$(\omega^2 - 15)(\omega^2 - 5) = 0 \Rightarrow \omega^2 = 15 \text{ or } \omega^2 = 5$$

For $\omega^2 = 15$. ($\omega = 3.873$, $T_P = 1.62$ s)

$$\begin{cases} (10-15)X_1 - 5X_2 = 0 \\ -5X_1 + (10-15)X_2 = 0 \end{cases} \Rightarrow X_1 = -X_2$$

For $\omega^2 = 5$ ($\omega = 2.236$, $T_P = 2.81$ s)

$$\begin{cases} (10-5)X_1 - 5X_2 = 0 \\ -5X_1 \quad (10-5)X_2 = 0 \end{cases} \Rightarrow X_1 = X_2.$$



(a) First mode (b) Second mode

The Power Method

- iterative method
- finds the largest eigenvalue

$$Ax = \lambda x$$

Example.

eigenvalue — eigenvector

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \Rightarrow \begin{aligned} 40x_1 - 20x_2 \qquad\quad &= \lambda x_1 \\ -20x_1 + 40x_2 - 20x_3 &= \lambda x_2 \\ -20x_2 + 40x_3 &= \lambda x_3 \end{aligned}$$

(1^{st} iteration) Guess $x = (1 \;\; 1 \;\; 1)$

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 20 \end{Bmatrix} = 20 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

(2^{nd} iteration)

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 40 \\ -40 \\ 40 \end{Bmatrix} = 40 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

(3^{rd} iteration)

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -80 \\ 60 \end{Bmatrix} = -80 \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix}$$

(4^{th} iteration)

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix} = \begin{Bmatrix} -50 \\ 70 \\ -50 \end{Bmatrix} = 70 \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix}$$

(5^{th} iteration)

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix} = \begin{Bmatrix} -48.5714 \\ 68.5714 \\ -48.5714 \end{Bmatrix} = 68.5714 \begin{Bmatrix} -0.70833 \\ 1 \\ -0.70833 \end{Bmatrix} \qquad \varepsilon = 2.08\%$$

Note. Eigenvalue: 68.28427
Eigenvector: (-0.707107 1 -0.707107)