10. LU Decomposition (Factorization) - Equivalent to Gaussian Elimination

$$A = L\,U$$

where
$$L = \begin{bmatrix} 1 & & & 0 \\ l_{21} & 1 & & \\ l_{31} & l_{32} & 1 & \\ \vdots & \vdots & & \ddots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ & u_{22} & u_{23} & \cdots & u_{2n} \\ & & u_{33} & \cdots & u_{3n} \\ & 0 & & \ddots & \vdots \\ & & & & u_{nn} \end{bmatrix}$$

Example

$$A\ x = b$$

$$\begin{array}{c} (-2) \\ (-\tfrac{1}{2}) \\ (1) \end{array} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

$$M_1 A x = M_1 b$$

where $M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{array}{c} \\ \\ (-3) \\ (\tfrac{1}{2}) \end{array} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

$$M_2(M_1 A x) = M_2(M_1 b)$$

where $M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}$

$$\begin{array}{c} \\ \\ \\ (-2) \end{array} \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -21 \end{bmatrix}$$

$$M_3(M_2 M_1 A x) = M_3(M_2 M_1 b)$$

where $M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}$

$$U \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

$$\underbrace{\underset{M}{\underline{M_3 M_2 M_1}}\,A}_{U}\ x = \underset{M}{\underline{M_3 M_2 M_1}}\ b$$

$$MA = U,$$

$$\underset{I}{(M^{-1} M)}\,A = M^{-1} U.$$

$$
\begin{aligned}
A &= M^{-1} U \\
&= (M_3 \cdot M_2 \cdot M_1)^{-1} U \\
&= \underbrace{M_1^{-1} \cdot M_2^{-1} \cdot M_3^{-1}}_{L} \cdot U
\end{aligned}
$$

$$
\begin{aligned}
L &= M_1^{-1} \cdot M_2^{-1} \cdot M_3^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & -\frac{1}{2} & 2 & 1 \end{bmatrix}
\end{aligned}
$$

We can obtain L by saving negative of multiplier during the elimination phase. Usually, we store those multipliers in situ. I.e.,

$$
\begin{pmatrix} U_{11} & U_{12} & U_{13} & U_{14} \\ l_{21} & U_{22} & U_{23} & U_{24} \\ l_{31} & l_{32} & U_{33} & U_{34} \\ l_{41} & l_{42} & l_{43} & U_{44} \end{pmatrix}
$$

Note. $l_{ii} = 1$ and they are not stored.

Question: Is there always LU-factorization for any non-singular matrix A?

Ex $\quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Ex.

$$\begin{array}{c} \\ \\ -2 \\ -\frac{1}{2} \end{array}\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & b \end{array} \\ \left(\begin{array}{ccc|c} 6 & -2 & 2 & 16 \\ 12 & -8 & 6 & 26 \\ 3 & -13 & 9 & -18 \end{array}\right) \end{array}$$

$$\begin{array}{c} \\ \\ -3 \end{array}\left(\begin{array}{ccc|c} 6 & -2 & 2 & 16 \\ 0 & -4 & 2 & -6 \\ 0 & -12 & 8 & -26 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 6 & -2 & 2 & 16 \\ 0 & -4 & 2 & -6 \\ 0 & 0 & 2 & -8 \end{array}\right)$$ **upper triangular form**

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

check:

$$L \cdot U = \begin{pmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 9 \end{pmatrix}$$

How to solve system of linear equations with LU-factorization

$$A \cdot x = b$$
$$\underline{(L \cdot U)\, x = b}$$

Solve (1) $L \cdot d = b$

(2) $U \cdot x = d$

Example. $\underset{A}{\begin{pmatrix} 6 & -2 & 2 \\ 12 & -8 & 6 \\ 3 & -13 & 9 \end{pmatrix}} \quad \underset{x}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \quad \underset{b}{\begin{pmatrix} 16 \\ 26 \\ -18 \end{pmatrix}}$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & 3 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\underset{L}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{1}{2} & 3 & 1 \end{pmatrix}} \underset{d}{\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}} \underset{b}{\begin{pmatrix} 16 \\ 26 \\ -18 \end{pmatrix}} \qquad d = \begin{pmatrix} 16 \\ -6 \\ -8 \end{pmatrix}$$

$$\underset{U}{\begin{pmatrix} 6 & -2 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix}} \underset{x}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \underset{d}{\begin{pmatrix} 16 \\ -6 \\ -8 \end{pmatrix}} \qquad x = \begin{pmatrix} 23/6 \\ -1/2 \\ -4 \end{pmatrix}$$

Note. Determinant

Cholesky Factorization (Decomposition)

$$A = U^T U$$

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2}$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}}{u_{ii}}, \quad j = i+1, \cdots n$$

Ex.

$$A = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}$$

$i=1$.

$$u_{11} = \sqrt{a_{11}} = \sqrt{6} = 2.44949$$

$$u_{12} = \frac{a_{12}}{u_{11}} = \frac{15}{2.44949} = 6.123724$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{15}{2.44949} = 22.45366$$

$i=2$.

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{55 - (6.123724)^2} = 4.1833$$

$$u_{23} = \frac{a_{23} - u_{12} \cdot u_{13}}{u_{22}} = \frac{225 - 6.123724(22.45366)}{4.1833} = 20.9165$$

$i=3$

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{979 - (22.45366)^2 - (20.9165)^2} = 6.110101$$

$$U = \begin{bmatrix} 2.44949 & 6.123724 & 22.45366 \\ 0 & 4.1833 & 20.9165 \\ 0 & 0 & 6.110101 \end{bmatrix}$$

Check.