12. Iterative Methods

- large sparse system

Iterative method starts from a first approximation which is successively improved until a sufficiently accurate solution is obtained.

$Ax = b$ and we start with x_i. Then

$$x_i = \frac{-\sum_{j=1, j\neq i}^{n} a_{ij} \cdot x_j + b_i}{a_{ii}} \qquad i = 1, 2, \dots, n$$

Jacobi's method computes a sequence of approximations $x^{(1)}, x^{(2)}, \dots$ by

$$x_i^{(k+1)} = \frac{-\sum_{j=1, j\neq i}^{n} a_{ij} \cdot x_j^{(k)} + b_i}{a_{ii}} \qquad i = 1, 2, \dots, n$$

Note. $x_i^{(k+1)}$ is not used until after a complete iteration.

Gauss-Seidel method

$$x_i^{(k+1)} = \frac{-\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} \cdot x_j^{(k)} + b_i}{a_{ii}} \qquad i = 1, 2, \dots, n$$

Note. $x_i^{(k+1)}$ is used as soon as it is computed.

$$x_i^{(k+1)} = x_i^{(k)} + \underbrace{\frac{-\sum_{j=1}^{i-1} a_{ij} \cdot x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij} \cdot x_j^{(k)} + b_i}{a_{ii}}}_{r_i^{(k)}}$$

SOR method

$$x_i^{(k+1)} = x_i^{(k)} + \omega r_i^{(k)} \qquad 0 < \omega < 2$$

Note. $\omega = 1 \Rightarrow$ Gauss-Seidel

Convergence.

$$\rho(A) = \max_{1 \le i \le n} |\lambda_i(A)| < 1$$

spectral radius

$$\|A_J\|_\infty = \max_{1 \le i \le n} \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{|a_{ij}|}{|a_{ii}|}$$

For diagonally dominant matrices, $\|A_J\|_\infty < 1 \Rightarrow$ converge.

Example.

$$A = \begin{pmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

$$X^{(0)} = (0.25, 0.5, 0, 0.25)^T$$

Jacobi's method

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$
1	0.25	0.5	0	0.25
2	0.375	0.625	0.125	0.375
3	0.4375	0.6875	0.1875	0.4375
4	0.46875	0.71875	0.21875	0.46875
5	0.48344	0.73438	0.23438	0.48344
6	0.49219	0.74172	0.24172	0.49219
7	0.49586	0.74609	0.24609	0.49586
8	0.49805	0.74793	0.24793	0.49805
⋮	...	...	...	...
∞	0.5	0.75	0.25	0.5

Gauss-Seidel's method

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$
1	0.25	0.5625	0.0625	0.40625
2	0.40625	0.70312	0.20312	0.47656
3	0.47656	0.73828	0.23828	0.49414
4	0.49414	0.74707	0.24707	0.49854
5	0.49854	0.74927	0.24927	0.49963
⋮	...	...	...	...

By a simple modification of Gauss-Seidel's method it is often possible to make a substantial improvement in the rate of convergence. We note that Eq. (5.6.2) can be written $x_i^{(k+1)} = x_i^{(k)} + r_i^{(k)}$, where $r_i^{(k)}$ is the current residual of the ith equation

$$r_i^{(k)} = \frac{-\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i}^{n} a_{ij}x_j^{(k)} + b_i}{a_{ii}}. \tag{5.6.8}$$

The iterative method,

$$x_i^{(k+1)} = x_i^{(k)} + \omega r_i^{(k)},$$

is the **successive overrelaxation (SOR) method**. Here ω, the relaxation parameter, should be chosen so that the rate of convergence is maximized. For $\omega = 1$, the method obviously reduces to Gauss-Seidel's method.

Example 5.6.2

The matrix A in Example 5.6.1 arises from the Laplace equation on a square with a mesh of $(N-1) \times (N-1)$ points where $N = 3$. A is positive-definite, and if we partition it—

$$A = \left(\begin{array}{c|cc|c} 4 & -1 & -1 & 0 \\ \hline -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ \hline 0 & -1 & -1 & 4 \end{array}\right)$$

—it is seen to be of the particular block-tridiagonal form in Theorem 5.6.3. It can be shown that for this model problem we have

$$\rho(B_J) = \cos\frac{\pi}{N}.$$

For $N = 3$ the formula (5.6.9) gives $\bar{\omega} = 1.0718$, and below we have solved $Ax = b$ using this optimal value in SOR.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$
1	0.26795	0.60770	0.07180	0.45002
2	0.43078	0.72828	0.23086	0.49264
3	0.49402	0.74798	0.24780	0.49940
4	0.49930	0.74980	0.24981	0.49994
...	...	...	...	...

The rate of convergence is evidently much better than with Gauss-Seidel's method, which corresponds to $\omega = 1$.