# 14 Polynomial Regression

$$y = a_0 + a_1 x + a_2 x^2 + e$$

$$\phi_r = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$$

$$\frac{\partial \phi}{\partial a_0} = -2 \sum (y_i - a_0 - a_1 x_i - a_2 x_i^2) = 0$$

$$\frac{\partial \phi}{\partial a_1} = -2 \sum x_i (y_i - a_0 - a_1 x_i - a_2 x_i^2) = 0$$

$$\frac{\partial \phi}{\partial a_2} = -2 \sum x_i^2 (y_i - a_0 - a_1 x_i - a_2 x_i^2) = 0$$

$$\begin{cases} n \cdot a_0 + (\sum x_i) a_1 + (\sum x_i^2) a_2 = \sum y_i \\ (\sum x_i) a_0 + (\sum x_i^2) a_1 + (\sum x_i^3) a_2 = \sum x_i y_i \\ (\sum x_i^2) a_0 + (\sum x_i^3) a_1 + (\sum x_i^4) a_2 = \sum x_i^2 y_i \end{cases}$$

Ex. 14.1

| x | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| y | 2.1 | 7.7 | 13.6 | 27.2 | 40.9 | 61.1 |

$$y = a_0 + a_1 x + a_2 x^2$$

$n=6$ $\quad \Sigma x_i = 15$ $\quad \Sigma x_i^2 = 55$ $\quad \Sigma y_i = 152.6$

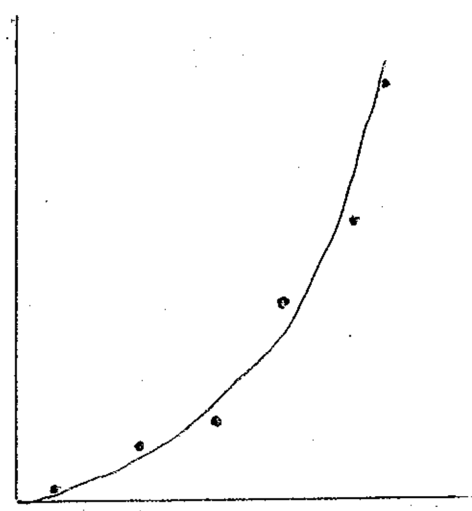
$\Sigma x_i = 15$ $\quad \Sigma x_i^2 = 55$ $\quad \Sigma x_i^3 = 225$ $\quad \Sigma x_i y_i = 585.6$

$\Sigma x_i^2 = 55$ $\quad \Sigma x_i^3 = 225$ $\quad \Sigma x_i^4 = 979$ $\quad \Sigma x_i^2 y_i = 2488.8$

$$\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 152.6 \\ 585.6 \\ 2488.8 \end{bmatrix}$$

$a_0 = 2.4786$
$a_1 = 2.3593$
$a_2 = 1.8607$

$$\therefore \; y = 2.4786 + 2.3593\,x + 1.8607\,x^2$$



## Non-polynomial Regression

Ex $\quad y = a e^{x^2} + b x^3 \qquad$ basis function $\equiv e^{x^2}, x^3$

| x | -1 | 0 | 1 |
|---|---|---|---|
| y | 0 | 1 | 2 |

(Solution) $\quad \phi(a,b) = \sum_{k=1}^{3} \left(a e^{x_k^2} + b x_k^3 - y_k\right)^2$

$$= (ae-b)^2 + (a-1)^2 + (ae+b-2)^2$$

$$\begin{cases} \dfrac{\partial \phi}{\partial a} = 2(ae-b)e + 2(a-1) + 2(ae+b-2)\cdot e = 0 & \text{—— ①} \\[2ex] \dfrac{\partial \phi}{\partial b} = -2(ae-b) + 2(ae+b-2) = 0 & \text{—— ②} \end{cases}$$

From ②, $\quad b = 1.$

plug into ①, $\quad 2(ae-1)e + 2(a-1) + 2(ae-1)e = 0$

$$4(ae-1)e + 2(a-1) = 0$$

$$4ae^2 - 4e + 2a - 2 = 0$$

$$(4e^2+2)a = 4e+2$$

$$\therefore a = \frac{2e+1}{2e^2+1}$$

$$y = \frac{2e+1}{2e^2+1} \cdot e^{x^2} + x^3$$

## Multiple Linear Regression

- two or more independent variables

$$y = a_0 + a_1 \underline{x_1} + a_2 \underline{x_2} + e$$

$$\phi_r = \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i}\right)^2$$

$$\frac{\partial \phi}{\partial a_0} = -2 \sum \left(y_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i}\right) = 0$$

$$\frac{\partial \phi}{\partial a_1} = -2 \sum x_{1,i} \left(y_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i}\right) = 0$$

$$\frac{\partial \phi}{\partial a_2} = -2 \sum x_{2,i} \left(y_i - a_0 - a_1 x_{1,i} - a_2 x_{2,i}\right) = 0$$

$$\begin{bmatrix} n & \sum x_{1,i} & \sum x_{2,i} \\ \sum x_{1,i} & \sum x_{1,i}^2 & \sum x_{1,i} \sum x_{2,i} \\ \sum x_{2,i} & \sum x_{1,i} \sum x_{2,i} & \sum x_{2,i}^2 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_{1,i} y_i \\ \sum x_{2,i} \cdot y_i \end{bmatrix}$$

Ex. 14.2

EXAMPLE 14.2 Multiple Linear Regression

From $y = 5 + 4x_1 - 3x_2$

| $x_1$ | $x_2$ | $y$ |
|---|---|---|
| 0 | 0 | 5 |
| 2 | 1 | 10 |
| 2.5 | 2 | 9 |
| 1 | 3 | 0 |
| 4 | 6 | 3 |
| 7 | 2 | 27 |

$$\begin{bmatrix} n & \sum x_{1,i} & \sum x_{2,i} \\ \sum x_{1,i} & \sum x_{1,i}^2 & \sum x_{1,i}x_{2,i} \\ \sum x_{2,i} & \sum x_{1,i}x_{2,i} & \sum x_{2,i}^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_{1,i}y_i \\ \sum x_{2,i}y_i \end{Bmatrix}$$

| $y$ | $x_1$ | $x_2$ | $x_1^2$ | $x_2^2$ | $x_1x_2$ | $x_1y$ | $x_2y$ |
|---|---|---|---|---|---|---|---|
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 2 | 1 | 4 | 1 | 2 | 20 | 10 |
| 9 | 2.5 | 2 | 6.25 | 4 | 5 | 22.5 | 18 |
| 0 | 1 | 3 | 1 | 9 | 3 | 0 | 0 |
| 3 | 4 | 6 | 16 | 36 | 24 | 12 | 18 |
| 27 | 7 | 2 | 49 | 4 | 14 | 189 | 54 |
| 54 | 16.5 | 14 | 76.25 | 54 | 48 | 243.5 | 100 |

$$\begin{bmatrix} 6 & 16.5 & 14 \\ 16.5 & 76.25 & 48 \\ 14 & 48 & 54 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 54 \\ 243.5 \\ 100 \end{Bmatrix}$$

$a_0 = 5$

$a_1 = 4$

$a_2 = -3$

# QR Factorization

- stable than normal equation method
- Matlab

Idea:

1. $\underset{m \geq n}{A_{m \times n}} = Q_{m \times n} \cdot R$

   Q: orthonormal colums (i.e. $Q^T \cdot Q = I$, $Q^T = Q^{-1}$)

   R: upper-triangular matrix with $r_{ii} > 0$.

2. $y = Q^T b$

3. $R \cdot x = y$

Note. <u>Modified Gram-Schmidt Method</u>