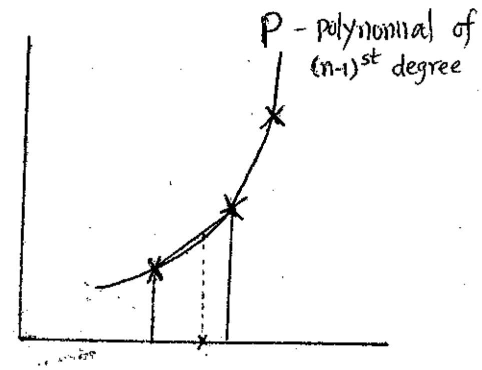
Polynomial Interpolation

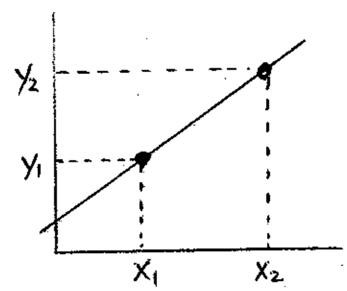
Observed data

x	x_1 x_2 ... x_n
y	y_1 y_2 ... y_n



We want to find a polynomial P(x) such that $P(x_i) = y_i$

Special case (n = 2)



$$P(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

check: $P(x_1) = y_1$
$P(x_2) = y_2$

Example.

x	0	2	3	4
f(x)	7	11	28	63

$$f(x) = a + bx + cx^2 + dx^3$$

or

$$\boxed{f(x) = ax^3 + bx^2 + cx + d} \qquad // \text{ Matlab}$$

Solution.

$$\begin{aligned} 7 &= d \\ 11 &= a(2)^3 + b(2)^2 + c(2) + d \\ 28 &= a(3)^3 + b(3)^2 + c(3) + d \\ 63 &= a(4)^3 + b(4)^2 + c(4) + d \end{aligned}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \\ 64 & 16 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 17 \\ 11 \\ 28 \\ 63 \end{bmatrix}$$

$a = 1, \; b = 0, \; c = -2, \; d = 7$

$$\underline{f(x) = x^3 - 2x + 7}$$

Note. $k(A) = 366.3624$

A is a Vandermonde matrix. → ill-conditioned.

See Example 15.1

Newton Algorithm for polynomial interpolation

- x_i are all distinct.

Suppose we included k points in P(x) such that $P(x_i) = y_i$, $1 \le i \le k$.
Assume we add another term to P(x) so that the new polynomial to reproduce one more entry in the table.

Consider $P(x) + c\,(x - x_1)(x - x_2) \ldots (x - x_k)$

We want to include (x_{k+1}, y_{k+1}) such that $P(x_{k+1}) = y_{k+1}$

$$P(x_{k+1}) + c\,(x_{k+1} - x_1)(x_{k+1} - x_2) \ldots (x_{k+1} - x_k) = y_{k+1}$$

Find c.

Note. Inductive reasoning

Example

x	0	2	3	4
y	7	11	28	63

Solution

$P_0(x) = 7$

$$P_1(x) = P_0(x) + c(x - x_1)$$
$$= 7 + cx \quad \Rightarrow c = 2$$
$$= 7 + 2x$$

$$P_2(x) = P_1(x) + c(x-x_1)(x-x_2)$$
$$= 7 + 2x + c\,x(x-2) \quad \Rightarrow 28 = 7 + 6 + 3c \Rightarrow c = 5$$
$$= 7 + 2x + 5x(x-2)$$

$$P_3(x) = P_2(x) + c(x-x_1)(x-x_2)(x-x_3)$$
$$= 7 + 2x + 5x(x-2) + c\,x(x-2)(x-3) \Rightarrow 63 = 55 + 8c \Rightarrow c = 1$$
$$= 7 + 2x + 5x(x-2) + x(x-2)(x-3)$$

Nested form: $7 + x[2 + (x-2)[5 + (x-3)]]$

Newton form: $p(x) = a_1 + a_2[(x-x_1)] + a_3[(x-x_1)(x-x_2)] + \cdots + a_n[(x-x_1)(x-x_2)\cdots(x-x_{n-1})]$

$$p(x) = a_1 + (x-x_1)(a_2 + (x-x_2)(a_3 + (x-x_3)(a_4 + \cdots + (x-x_{n-1})a_n))\cdots)$$
$$= (\cdots((a_n(x-x_{n-1}) + a_{n-1})(x-x_{n-2}) + a_{n-2})\cdots)(x-x_1) + a_1$$

Computer program to find $P(t)$

$v_1 = a_n$

$v_2 = v_1(t - x_{n-1}) + a_{n-1}$

$v_3 = v_2(t - x_{n-2}) + a_{n-2}$

$\vdots$

$v_n = v_{n-1}(t - x_1) + a_1$

$\Rightarrow$

```
V = A(n)
for i := (n-1) downto 1
    V = V * (t - X_i) + A(i)
endfor
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Note. $P(t) = v_n$.

Calculating Coefficients a_i using Divided Differences

x	x_1	x_2	x_3	$\cdots$	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\cdots$	$f(x_n)$

Newton form:

$$P_{n-1}(x) = \overbrace{a_1 + a_2(x-x_1) + a_3(x-x_1)(x-x_2) + \cdots +}^{P_{n-2}(x)} a_n(x-x_1)\cdots(x-x_n)$$

$$= P_{n-2}(x) + a_n(x-x_1)\cdots(x-x_n)$$

Compact form: $$P_{n-1}(x) = \sum_{i=1}^{n} a_i \prod_{j=1}^{i-1}(x-x_j)$$

Observation: P_{n-1} is obtained from P_{n-2} by adding one more term, without altering the coefficients already present in P_{n-2} itself.

To determine a_i, we solve

$$\begin{cases} f(x_1) = a_1 \\ f(x_2) = a_1 + a_2(x_2-x_1) \\ f(x_3) = a_1 + a_2(x_3-x_1) + a_3(x_3-x_1)(x_3-x_2) \\ \vdots \\ \boxed{f(x_k) = \sum_{i=1}^{k} a_i \prod_{j=1}^{i-1}(x_k-x_j) \quad k=1,2,\cdots n} \end{cases} \Rightarrow \begin{aligned} a_1 &= f(x_1) &&= f[x_1] \\ a_2 &= \frac{f(x_2)-f(x_1)}{x_2-x_1} &&= f[x_1,x_2] \\ a_3 &= \frac{f(x_3)-a_1-a_2(x_3-x_1)}{(x_3-x_1)(x_3-x_2)} &&= f[x_1,x_2,x_3] \\ &\vdots && \vdots \\ a_k &= \frac{f(x_k)-\sum_{i=1}^{k-1} a_i \prod_{j=1}^{i-1}(x_k-x_j)}{\prod_{j=1}^{k-1}(x_k-x_j)} &&= f[x_1,x_2\cdots x_k] \end{aligned}$$

I.e. $a_k = \underline{f[x_1, x_2 \cdots x_k]}$

↑ divided differences of f

<u>Note</u>. $\prod_{j=1}^{0}(x-x_j) = 1$

Theorem on Recursive Property of Divided Differences

$$f[x_1, x_2, \cdots x_n] = \frac{f[x_2 \cdots x_n] - f[x_1 \cdots x_{n-1}]}{x_n - x_1}$$

In general,

$$f[x_i, x_{i+1} \cdots x_{j-1}, x_j] = \frac{f[x_{i+1}, x_{i+2}, \cdots x_j] - f[x_i, x_{i+1}, \cdots x_{j-1}]}{x_j - x_i}$$

Note. Invariance Theorem.

$f[x_1, x_2 \cdots x_k]$ is invariant under all permutations of $x_1, x_2, \cdots x_k$.

Ex.

$$f[x_i] = f(x_i)$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Divided Difference Table

x	$f[\ ]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$
x_1	$f[x_1]$			
		$f[x_1, x_2]$		
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$
x_3	$f[x_3]$		$f[x_2, x_3, x_4]$	
		$f[x_3, x_4]$		
x_4	$f[x_4]$			

$$P(x) = \underbrace{f[x_1]}_{a_1} + \underbrace{f[x_1, x_2]}_{a_2}(x - x_1) + \underbrace{f[x_1, x_2, x_3]}_{a_3}(x - x_1)(x - x_2) + \underbrace{f[x_1, x_2, x_3, x_4]}_{a_4}(x - x_1)(x - x_2)(x - x_3)$$

Ex.

X	1	-4	0
f(x)	3	13	-23

	X	f[ ]	f[,]	f[, ,]
x_1	1	(3)	(-2)	(7)
x_2	-4	13	-9	
x_3	0	-23		

Pyramid

a_1: $f[x_1] = f(x_1) = 3$

a_2: $f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1} = \dfrac{13-3}{-4-1} = -2$

$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2} = \dfrac{-23-13}{0+4} = -9$

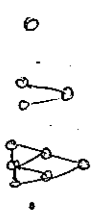
a_3: $f[x_1, x_2, x_3] = \dfrac{f[x_2,x_3] - f[x_1,x_2]}{x_3 - x_1} = \dfrac{-9+2}{0-1} = 7$

$\therefore$ $\underline{f(x) = 3 - 2(x-1) + 7(x-1)(x+4)}$ $\quad (= 7x^2 + 19x - 23)$

<u>Complexity</u>

• Division

n=1	0	
n=2	1	
n=3	1+2	
n=4	1+2+3	
⋮	⋮	⋮



$1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$

• Addition
2 × Divisions

$n(n-1)$.

(continued)

	x1	x2	x3
x	-4	0	1
f(x)	13	-23	3

	x	f[ ]	f[,]	f[, ,]
x1	-4	(13)		
			(-9)	
x2	0	-23		(7)
			26	
x3	1	3		

a1: $f[x1] = f(x1) = 13$

a2: $f[x1,x2] = \dfrac{f[x_2] - f[x1]}{x2 - x1} = \dfrac{-23 - 13}{0 - (-4)} = \dfrac{-36}{4} = -9$

$f[x2,x3] = \dfrac{f[x_3] - f[x2]}{x3 - x2} = \dfrac{3 - (-23)}{1 - 0} = \dfrac{26}{1} = 26$

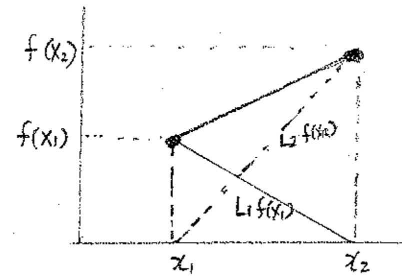
a3: $f[x1,x2,x3] = \dfrac{f[2,3] - f[1,2]}{x3 - x1} = \dfrac{26 - (-9)}{1 - (-4)} = \dfrac{35}{5} = 7$

$\therefore\ f(x) = 13 - 9(x+4) + 7(x+4)x \qquad (= 7x^2 + 19x - 23)$

Lagrange Interpolating Polynomial

Cardinal functions, $l_1, l_2, \cdots l_n$

$$l_i(x_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$



Idea:

$$\boxed{P_{n-1}(x) = \sum_{i=1}^{n} f(x_i) \cdot l_i(x)}$$

Put $x = x_j$

$$P_{n-1}(x_j) = \sum_{i=1}^{n} f(x_i) \cdot l_i(x_j) = f(x_j) \cdot \underbrace{l_j(x_j)}_{=1} = f(x_j)$$

I.e. P_{n-1} is the interpolating polynomial for f at nodes $x_1, x_2, \cdots x_n$.

$$\boxed{l_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right) \qquad 1 \leq i \leq n}$$

Example

x	1	-4	0
$f(x)$	3	13	-23

Solution.

$$l_1(x) = \frac{(x+4) \cdot x}{(1+4) \cdot 1} = \frac{1}{5} x (x+4)$$

$$l_2(x) = \frac{(x-1) \cdot x}{(-4-1)(-4)} = \frac{1}{20} x (x-1)$$

$$l_3(x) = \frac{(x-1)(x+4)}{(0-1)(0+4)} = -\frac{1}{4} (x-1)(x+4)$$

$$\therefore P_2(x) = \frac{3}{5} x (x+4) + \frac{13}{20} x (x-1) + \frac{23}{4} (x-1)(x+4)$$

Inverse Interpolation

Ex.

x	1	2	3	4	5	6	7
f(x)	1	0.5	0.3333	0.25	0.2	0.1667	0.1429

$f(x) = \frac{1}{x}$

Find x corresponding to $f(x) = 0.3$. ($X_{true} = 3.3333$)

Method 1.

Switch roles (x ⟺ f(x)) and perform polynomial interpolation.

Problem.

1. not evenly spread
2. oscillation

Method 2.

polynomial interpolation with 3 points

f(x)	0.25	0.3333	0.5
x	4	3	2

$$f(x) = 0.041667\,x^2 - 0.375\,x + 1.08333$$

Original problem ➔ root finding problem: $f(x) = 0.3$

$$0.041667\,x^2 - 0.375\,x + 1.08333 = 0.3$$

x1 = 5.504147

x2 = <u>3.295842</u> ← (*)

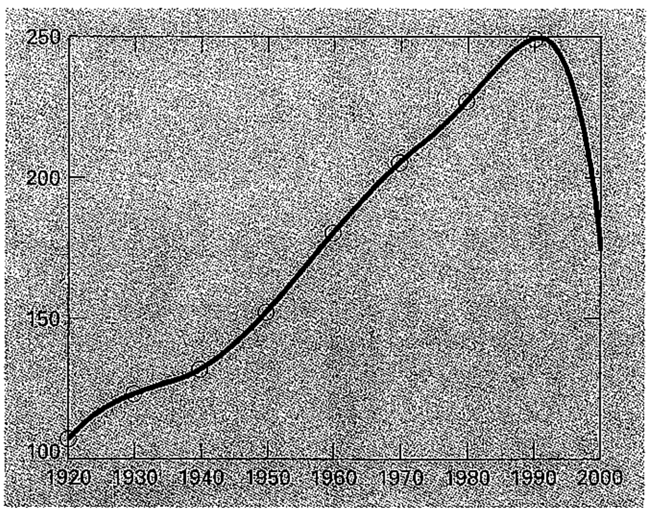
Danger of Extrapolation

	1	2	3	4	5	6	7	8	
Year	1920	1930	1940	1950	1960	1970	1980	1990	*2000*
Population	106	123	132	152	181	295	227	249	

Polynomial interpolation with 8 data points ➔ 7^{th} order polynomial

Prediction for year 2000 : 175

Extrapolation → Erroneous prediction



Oscillation

Runge's Function: $f(x) = \dfrac{1}{1+25x^2}$

Serpentine Curve: $f(x) = \dfrac{x}{\frac{1}{4} + x^2}$