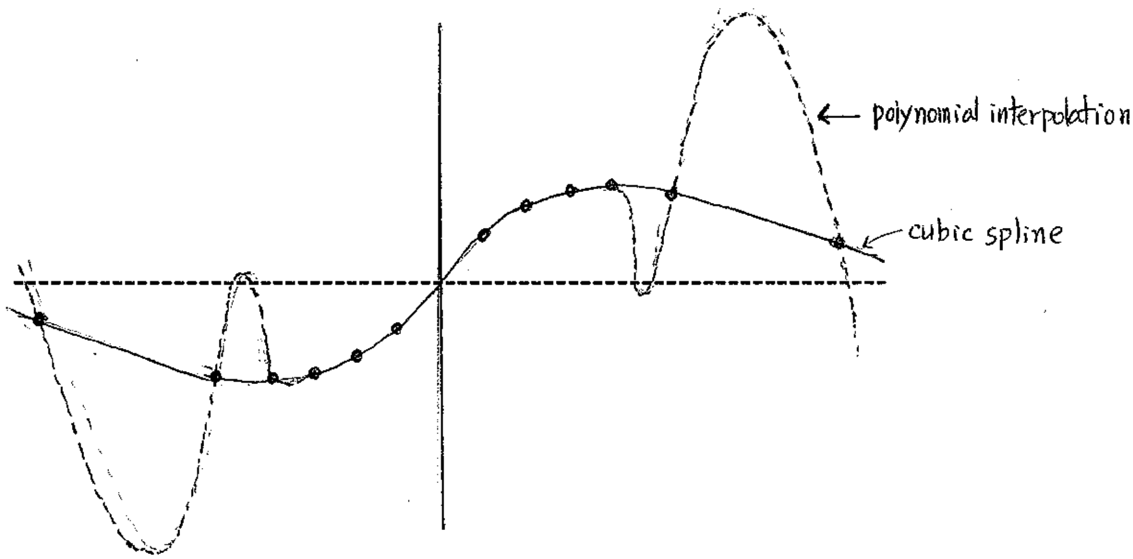
Spline Functions

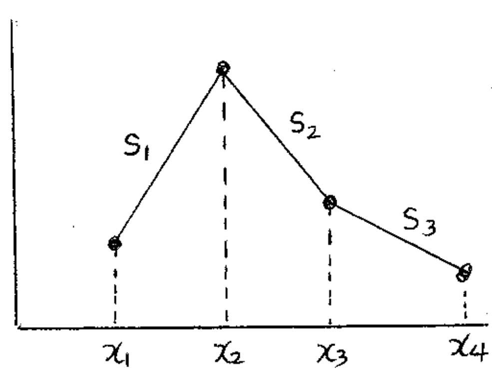
- another approximation

Motivation.

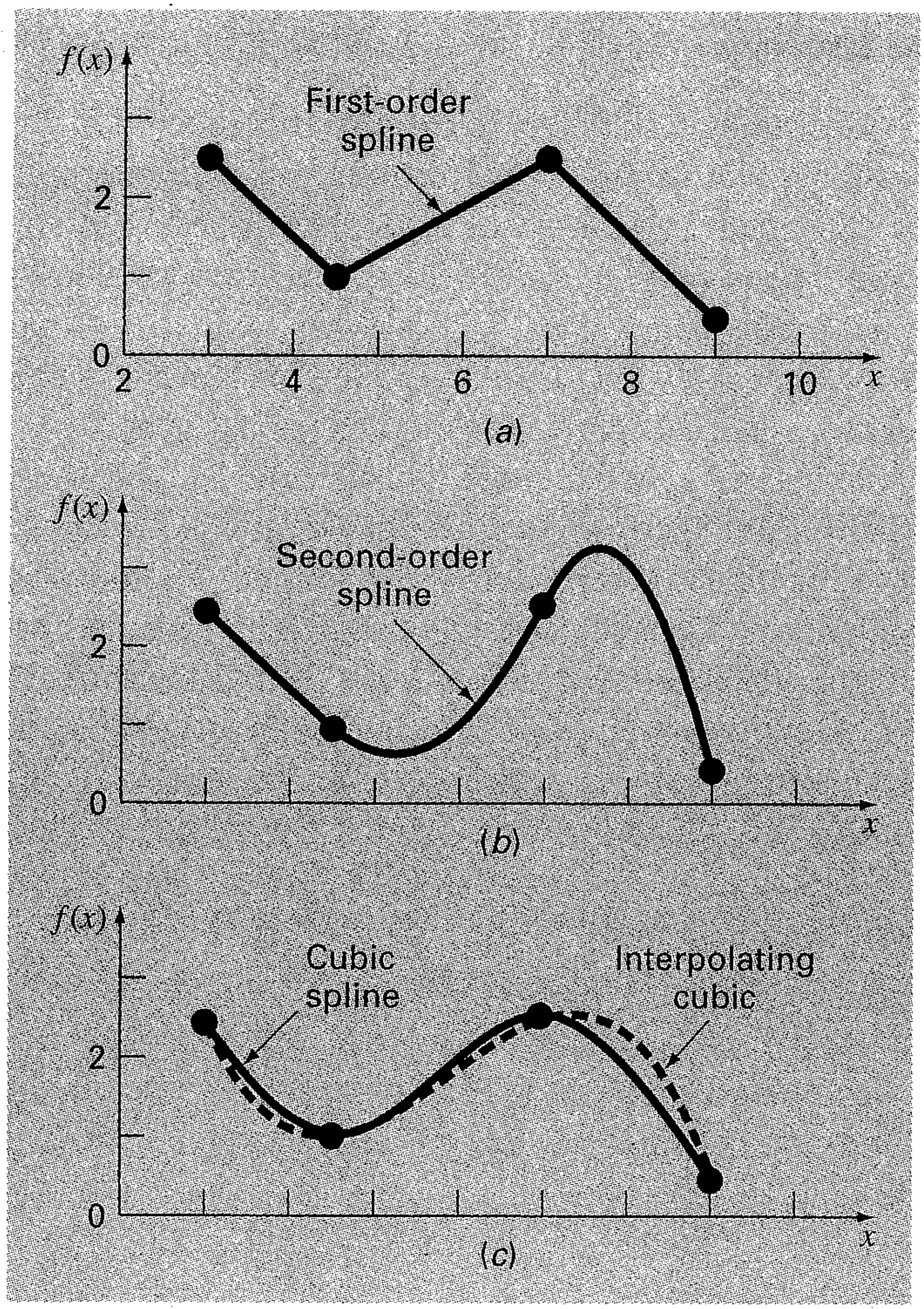
Serpentine Curve: $y = \dfrac{x}{\frac{1}{4} + x^2}$



Linear Splines (First-order Splines)



$$S_i(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i)$$

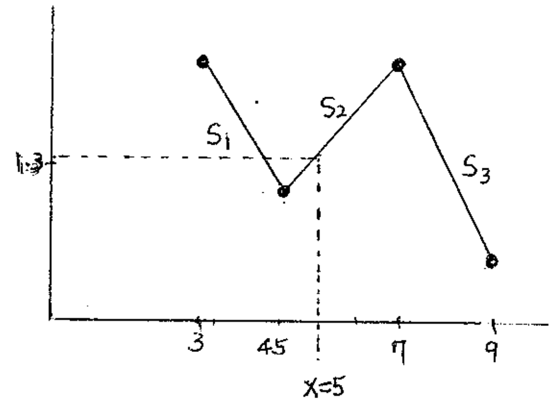


f(x)
First-order spline
2
0
2
4
6
8
10
x
(a)
f(x)
Second-order spline
2
0
x
(b)
f(x)
Cubic spline
Interpolating cubic
2
0
x
(c)

FIGURE 16.4

Ex 16.1 Find f(5).

i	X_i	$f(X_i)$
1	3.0	2.5
2	4.5	1.0
3	7.0	2.5
4	9.0	0.5



x = 5 falls into the second interval.

$$S_2(x) = 1.0 + \frac{2.5 - 1.0}{7.0 - 4.5}(x - 4.5)$$

For x = 5,

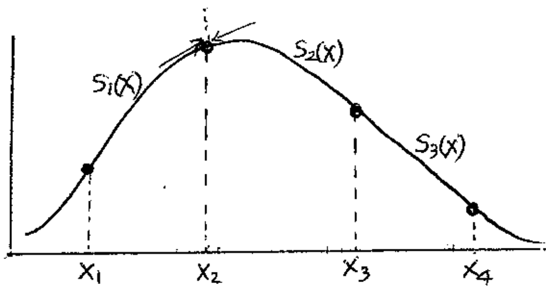
$$S_2(5) = 1.0 + \frac{2.5 - 1.0}{7.0 - 4.5}(5 - 4.5) = 1.3$$

Note. Slope changes abruptly at the knots.

Application. Table look-up

Quadratic Splines (Second-degree Splines)

- continuous first derivative



$$S_i(x) = \begin{cases} S_1(x) & x_1 \le x \le x_2 \\ S_2(x) & x_2 \le x \le x_3 \\ S_3(x) & x_3 \le x \le x_4 \\ \vdots & \vdots \end{cases}$$

$$Si(x) = \underline{\underline{a_i}} + \underline{\underline{b_i}}(x - x_i) + \underline{\underline{c_i}}(x - x_i)^2 \quad \text{.........} (1)$$

1. Continuous condition

   $f_i = a_i + b_i(x_i - x_i) + c_i(x_i - x_i)^2$

   $f_i = a_i$ .... (2) // (n-1) a_i's

   (2) → (1)

   $S_i(x) = f_i + b_i(x - x_i) + c_i(x - x_i)^2$

2. Function values of adjacent polynomials must be equal at the knot.

   $f_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 = f_{i+1} + b_{i+1}(x_{i+1} - x_{i+1}) + c_{i+1}(x_{i+1} - x_{i+1})^2$

   Let $h_i = x_{i+1} - x_i$ Then

   $f_i + b_i h_i + c_i h_i^2 = f_{i+1}$ .... (3) // (n-1) equations

3. Continuity of derivations:

   First derivatives at the internal knots must be equal (for smooth joining).

   Differenciate (1),

   $S_i'(x) = b_i + 2c_i(x - x_i)$

   $b_i + 2c_i h_i = b_{i+1}$ // (n-2) equations

4. One more condition is needed.

   Choose $S_i''(x) = 0$ → $c_1 = 0$ // one condition

Ex.

i	x_i	f_i	h_i
1	3	2.5	1.5
2	4.5	1.0	2.5
3	7	2.5	2
4	9	0.5	

4 Knots, 3 intervals. $S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2$

Continuity condition with $c_1 = 0$

$$f_1 + b_1 h_1 = f_2 \quad 2.5 + 1.5 b_1 = 1.0 \Rightarrow 1.5 b_1 = -1.5$$

$$f_2 + b_2 h_2 + c_2 h_2^2 = f_3 \quad 1.0 + 2.5 b_2 + (2.5)^2 c_2 = 2.5 \Rightarrow 2.5 b_2 + 6.25 c_2 = 1.5$$

$$f_3 + b_3 h_3 + c_3 h_3^2 = f_4 \quad 2.5 + 2 \cdot b_3 + (2)^2 c_3 = 0.5 \Rightarrow 2 b_3 + 4 c_3 = -2$$

Continuity of derivatives with $c_1 = 0$

$$b_1 = b_2 \Rightarrow b_1 - b_2 = 0$$

$$b_2 + 2 c_2 h_2 = b_3 \quad b_2 + 2 \cdot (2.5) c_2 = b_3 \Rightarrow b_2 - 5 c_2 - b_3 = 0$$

$f_1 = 2.5,\ f_2 = 1.0,\ f_3 = 2.5,\ f_4 = 0.5,\ h_1 = 1.5,\ h_2 = 2.5\ h_3 = 2.$

$$\begin{bmatrix} 1.5 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 6.25 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -1 & 0 \end{bmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ c_2 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1.5 \\ 1.5 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

$b_1 = -1,\ b_2 = -1,\ b_3 = 2.2.\ c_2 = 0.64,\ c_3 = -1.6$

$$\therefore \begin{cases} S_1(x) = 2.5 - (x-3) \\ S_2(x) = 1.0 - (x-4.5) + 0.64(x-4.5)^2 \\ S_3(x) = 2.5 + 2.2(x-7.0) - 1.6(x-7.0)^2 \end{cases}$$

$x = 5, \quad S_2(5) = 1.0 - (5-4.5) + 0.64(5-4.5)^2 = 0.66$

Cubic Splines

– most popular

$$\boxed{S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3} \quad \cdots\cdots (1)$$

n knots $\Rightarrow$ $(n-1)$ intervals $\Rightarrow$ $4(n-1)$ unknowns

Conditions

1. Spline must pass through all the data points.

$$f_i = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$$

$$\Rightarrow \quad a_i = f_i$$

$$\therefore \boxed{S_i(x) = f_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3} \quad \cdots\cdots (2)$$

2. Each cubics must join at the knots.

(for $(i+1)^{st}$ knot,)

$$\boxed{f_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = f_{i+1}} \quad \cdots\cdots (3)$$

3. The first derivative at the interior nodes must be equal : $(n-2)$

$$S_i'(x) = b_i + 2c_i(x-x_i) + 3d_i(x-x_i)^2 \quad \cdots\cdots (4)$$

$$\boxed{b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1}} \quad \cdots\cdots (5)$$

4. The second derivatives at the interior nodes must be equal : $(n-2)$

$$S_i''(x) = 2c_i + 6d_i(x-x_i) \quad \cdots\cdots (6)$$

$$2c_i + 6d_i h_i = 2c_{i+1} \quad \text{or}$$

$$\boxed{c_i + 3d_i \cdot h_i = c_{i+1}} \quad \cdots\cdots (7)$$

Solve (7) for d_i,

$$\boxed{d_i = \frac{c_{i+1} - c_i}{3h_i}} \quad \text{-------------- (8)}$$

Substitute d_i into (3),

$$f_i + b_i h_i + c_i h_i^2 + \frac{c_{i+1} - c_i}{3h_i} h_i^3 = f_{i+1}$$

$$\boxed{f_i + b_i h_i + \frac{h_i^2}{3}(2c_i + c_{i+1}) = f_{i+1}} \quad \text{---------- (9)}$$

Substitute d_i into (5),

$$b_{i+1} = b_i + 2c_i h_i + 3\frac{c_{i+1} - c_i}{3h_i} h_i^2$$

$$\boxed{b_{i+1} = b_i + h_i(c_i + c_{i+1})} \quad \text{-------------- (10)}$$

Solve (9) for b_i,

$$b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1}) \quad \text{---------- (11)}$$

change index i to $(i-1)$ in (11),

$$b_{i-1} = \frac{f_i - f_{i-1}}{h_{i-1}} - \frac{h_{i-1}}{3}(2c_{i-1} + c_i) \quad \text{-------- (12)}$$

Change index i to $(i-1)$ in (10)

$$b_i = b_{i-1} + h_{i-1}(c_{i-1} + c_i) \quad \text{-------------- (13)}$$

Plug b_i (11) and b_{i-1} (12) into (13),

$$\frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}(2c_i + c_{i+1}) = \frac{f_i - f_{i-1}}{h_{i-1}} - \frac{h_{i-1}}{3}(2c_{i-1} + c_i) + h_{i-1}(c_{i-1} + c_i)$$

$$\frac{f_{i+1} - f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}} = \frac{2}{3}h_i c_i + \frac{1}{3}h_i c_{i+1} - \frac{2}{3}h_{i-1}c_{i-1} - \frac{1}{3}h_{i-1}c_i + h_{i-1}c_{i-1} + h_{i-1}c_i$$

$$= \frac{1}{3}h_{i-1}c_{i-1} + \frac{2}{3}(h_i + h_{i-1})c_i + \frac{1}{3}h_i c_{i+1}$$

Multiply by 3 and swap,

$$h_{i-1}c_{i-1} + 2(h_{i-1} + h_i)c_i + h_i c_{i+1} = 3\frac{f_{i+1} - f_i}{h_i} - 3\frac{f_i - f_{i-1}}{h_{i-1}} \quad \text{--- (14)}$$

Note that difinite difference $f[x_i, x_j] \equiv \dfrac{f_i - f_j}{x_i - x_j}$.

$$\boxed{h_{i-1}c_{i-1} + 2(h_{i-1}+h_i)c_i + h_i c_{i+1} = 3\left(f[x_{i+1},x_i] - f[x_i,x_{i-1}]\right)}_{j=2,3,\dots,n-2} \quad \text{------ (15)}$$

Need two additional conditions.

<u>Natural spline</u> : second derivatives at two end knots are zero.

first node: $S_1''(x_0) = 0 = 2c_1 + 6\cdot d_1(x_1 - x_1) \Rightarrow c_1 = 0$

Last node: $S_{n-1}''(x_n) = 0 = 2\cdot c_{n-1} + 6 d_{n-1}(x - x_{n-1})$ ---------- (16)

Choose c_n such that (16) satifies. $\Rightarrow c_{n-1} + 3 d_{n-1} h_{n-1} = c_n = 0 \Rightarrow c_n = 0$

Procedure for cubic spline:

(1) Solve for c_i

$$\begin{bmatrix} 1 & & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & \ddots & \ddots & \ddots & \\ & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 3(f[x_3,x_2] - f[x_2,x_1]) \\ \vdots \\ 3(f[x_n - x_{n-1}] - f[x_{n-1},x_{n-2}]) \\ 0 \end{bmatrix}$$

(2) Solve (11) for b_i

(3) Solve (8) for d_i

<u>Note</u> $a_i = f_i$

EXAMPLE 16.3 Natural Cubic Splines

i	x_i	f_i
1	3.0	2.5
2	4.5	1.0
3	7.0	2.5
4	9.0	0.5

Solution. The first step is to employ Eq. (16.27) to generate the set of simultaneous equations that will be utilized to determine the c coefficients:

$$\begin{bmatrix} 1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & \\ & h_2 & 2(h_2+h_3) & h_3 \\ & & & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3(f[x_3,x_2]-f[x_2,x_1]) \\ 3(f[x_4,x_3]-f[x_3,x_2]) \\ 0 \end{Bmatrix}$$

The necessary function and interval width values are

$$\begin{aligned} f_1 &= 2.5 & h_1 &= 4.5-3.0=1.5 \\ f_2 &= 1.0 & h_2 &= 7.0-4.5=2.5 \\ f_3 &= 2.5 & h_3 &= 9.0-7.0=2.0 \\ f_4 &= 0.5 \end{aligned}$$

These can be substituted to yield

$$\begin{bmatrix} 1 & & & \\ 1.5 & 8 & 2.5 & \\ & 2.5 & 9 & 2 \\ & & & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4.8 \\ -4.8 \\ 0 \end{Bmatrix}$$

These equations can be solved using MATLAB with the results:

$$\begin{aligned} c_1 &= 0 & c_2 &= 0.839543726 \\ c_3 &= -0.766539924 & c_4 &= 0 \end{aligned}$$

Equations (16.21) and (16.18) can be used to compute the b's and d's

$$\begin{aligned} b_1 &= -1.419771863 & d_1 &= 0.186565272 \\ b_2 &= -0.160456274 & d_2 &= -0.214144487 \\ b_3 &= 0.022053232 & d_3 &= 0.127756654 \end{aligned}$$

These results, along with the values for the a's [Eq. (16.11)], can be substituted into Eq. (16.10) to develop the following cubic splines for each interval:

$$s_1(x) = 2.5 - 1.419771863(x-3) + 0.186565272(x-3)^3$$

$$s_2(x) = 1.0 - 0.160456274(x-4.5) + 0.839543726(x-4.5)^2 - 0.214144487(x-4.5)^3$$

$$s_3(x) = 2.5 + 0.022053232(x-7.0) - 0.766539924(x-7.0)^2 + 0.127756654(x-7.0)^3$$

The three equations can then be employed to compute values within each interval. For example, the value at $x = 5$, which falls within the second interval, is calculated as

$$s_2(5) = 1.0 - 0.160456274(5-4.5) + 0.839543726(5-4.5)^2 - 0.214144487(5-4.5)^3 = 1.102889734.$$

The total cubic spline fit is depicted in Fig. 16.4c.