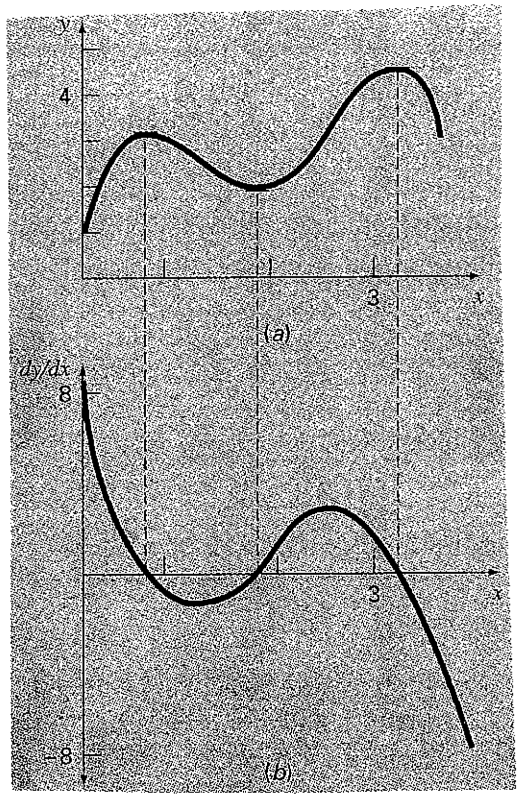
Ordinary Differential Equations

$y = f(x)$

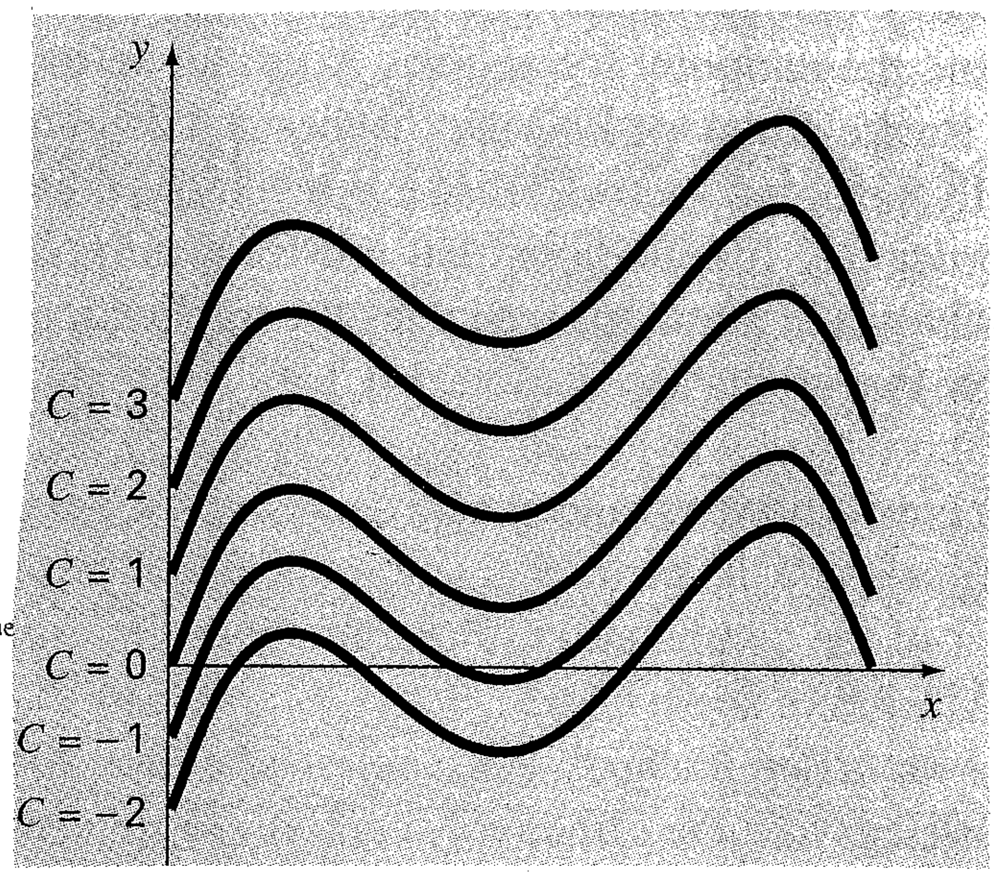
$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$

$\frac{dy}{dx} = f'(x)$

$y' = -2x^3 + 12x^2 - 20x + 8.5$



$y = \int (-2x^3 + 12x^2 - 20x + 8.5)\, dx = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + C$



initial value

Ordinary Differential Equations

Equations	Solution
$x' - x = e^t$	$x(t) = t\,e^t + c \cdot e^t$
$x'' + 9x = 0$	$x(t) = c_1 \sin 3t + c_2 \cos 3t$
$x' + \frac{1}{2x} = 0$	$x(t) = \sqrt{c - t}$

Initial-Value Problem

$$\begin{cases} x' = f(t, x) & // \ \frac{dx(t)}{dt} = f(t, x(t)) \ // \\ x(a) \text{ given.} & \text{- Initial condition} \end{cases}$$

Examples.

$$\begin{cases} x' = x + 1 \\ x(0) = 0 \end{cases} \quad \Rightarrow \quad x = e^t - 1$$

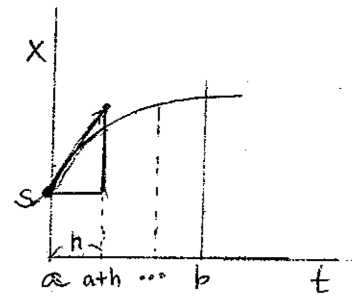
$$\begin{cases} x' = 6t - 1 \\ x(1) = 6 \end{cases} \quad \Rightarrow \quad x = 3t^2 - t + 4$$

$$\begin{cases} x' = \frac{t}{x+1} \\ x(0) = 0 \end{cases} \quad \Rightarrow \quad x = \sqrt{t^2 + 1} - 1$$

Euler's Method — Taylor-series method of order 1.

$$\begin{cases} x' = f(t, x(t)) & [a, b] \\ x(a) = s \end{cases}$$

$$x(t+h) \cong x(t) + h \cdot x'(t)$$
$$= x(t) + h\, f(t, x(t))$$



Note. Recurrence relation

$$h = \frac{(b-a)}{n}$$

<u>Procedure Euler</u>

$h \leftarrow \frac{b-a}{n}$

$x \leftarrow s$

$t \leftarrow a$

for $k \leftarrow 1$ to n

  $x \leftarrow x + h \cdot f(t,x)$

  $t \leftarrow t + h$

end

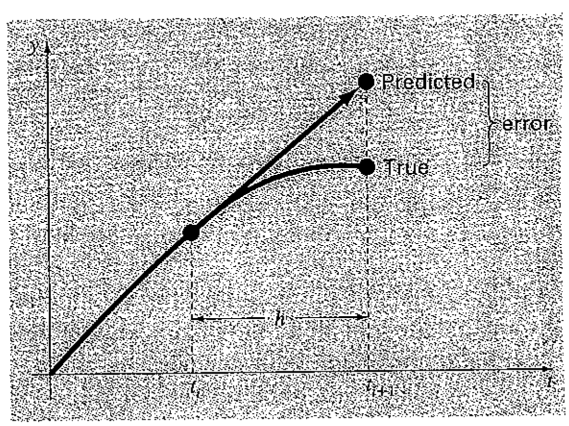


FIGURE 20.1
Euler's method.

Stability of Euler's Method

$$\begin{cases} dy/dt = -ay \\ y(0) = y_0 \end{cases}$$

[Analytic]

$$y = y_0 e^{-at}$$

Note. $\lim_{t \to \infty} (y_0 e^{-at}) = 0$

[Numerical]

$$y_{i+1} = y_i + \frac{dy_i}{dt} h$$

$$= y_i - a y_i h$$

$$= y_i \underbrace{(1 - ah)}_{\text{amplification factor}}$$

To converge to zero,

$$|1 - ah| < 1 \quad \Rightarrow \quad h < \frac{2}{a}$$

Conditionally stable.

Taylor series method of order 4

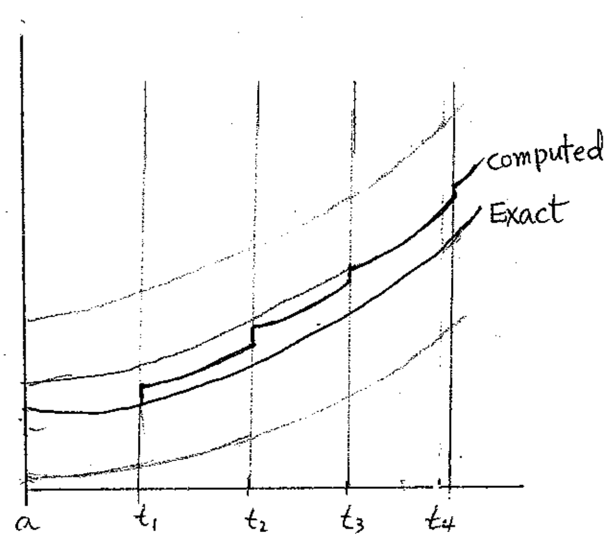
Ex. $\begin{cases} x' = 1 + x^2 + t^3 \\ x(0) = 0 \end{cases}$

What is $x(1)$?

$$x' = 1 + x^2 + t^3$$
$$x'' = 2x \cdot x' + 3t^2$$
$$x''' = 2x \cdot x'' + 2x' \cdot x' + 6t$$
$$x^{iv} = 2x \cdot x''' + 6x' x'' + 6$$

$$\boxed{x \leftarrow \underbrace{x + h x'}_{\text{Euler's}} + \underbrace{\frac{h^2}{2} x'' + \frac{h^3}{6} x''' + \frac{h^4}{24} x^{iv}}_{\substack{\text{Truncation error} \\ O(h^2)}}}$$

Note. Error: truncation error + round-off error
Error propagate each iteration



$$\begin{cases} x' = 1 + x^2 + t^3 \\ x(0) = 0 \end{cases} \quad \rightarrow x(1)$$

```
c     program ode
      call euler
      call taylor4
      stop
      end
c
      subroutine euler
      data a/0.0/, b/1.0/, s/0.0/
      data n/128/
      f(t,x) = 1.0 + x*x + t**3
      h = (b-a) / real(n)
      x = s
      t = a
      print 5, t, x
      do 10 k = 1,n
         x = x + h * f(t,x)
         t = t + h
         print 5, t, x
   10 continue
    5 format(3x, f10.6, 2x, f10.6)
      return
      end
c
      subroutine taylor4
      data t,x/2*0.0/
      data h/7.8125e-3/
      print 5, t, x
      do 10 k = 1,128
         x1 = 1.0 + x*x + t**3
         x2 = 2.0 * x * x1 + 3.0 * t * t
         x3 = 2.0 * x * x2 + 2.0 * x1 * x1 + 6.0 * t
         x4 = 2.0 * x * x3 + 6.0 * x1 * x2 + 6.0
         x = x + h* (x1 + h * (x2/2.0 + h * (x3/6.0 + h * x4/24.0)))
         t = real(k) * h
         print 5, t, x
   10 continue
    5 format(5x, f10.6, 3x, f10.6)
      return
      end
```

x(t) value at t =1

n	Euler's	Taylor's Order 4
1	1.0	1.0
2	1.1875	1.868390
4	1.412197	1.969476
8	1.614874	1.989215
16	1.767678	1.991570
32	1.867413	1.991776
64	1.925863	1.991792
128	1.957787	1.991792
	$O(h^2)$	$O(h^5)$

Euler Method with Repeated Richardson Extrapolation

$$\frac{\Delta}{1},\ \frac{\Delta}{3}\quad \frac{\Delta}{7}\quad \frac{\Delta}{15}\quad \frac{\Delta}{31}\quad \cdots \quad \frac{1}{2^n-1}$$

Ex. $\begin{cases} x' = 1 + x^2 + t^3 \\ x(0) = 0 \end{cases}$ $\qquad\qquad x(1)?$

$h = 1 \qquad x(\overset{t}{0} + \overset{h}{1}) \simeq x(0) + 1\cdot(1 + 0^2 + 0^2) = \underline{1}$

$h = ½ \qquad x(0 + ½) \simeq x(0) + ½\,(1 + 0^2 + 0^2) = 0.5$

$\qquad\qquad x(½ + ½) = x(½) + ½\,(1 + 0.5^2 + (½)^2) = \underline{1.25}$

$h = ¼ \qquad x(0 + ¼) \simeq x(0) + ¼\,(1 + 0^2 + 0^2) = 0.25$

$\qquad\qquad x(\tfrac{1}{4} + \tfrac{1}{4}) = x(\tfrac{1}{4}) + \tfrac{1}{4}\,(1 + 0.25^2 + (\tfrac{1}{4})^2) = 0.53125$

$\qquad\qquad x(\tfrac{1}{2} + \tfrac{1}{4}) = x(\tfrac{1}{2}) + \tfrac{1}{4}\,(1 + 0.53125^2 + (\tfrac{1}{2})^2) = 0.9143066$

$\qquad\qquad x(\tfrac{3}{4} + \tfrac{1}{4}) \simeq x(\tfrac{3}{4}) + \tfrac{1}{4}\,(1 + 0.9143066^2 + (\tfrac{3}{4})^2) = \underline{1.5139207}$

h	f		$\frac{\Delta}{1}$		$\frac{\Delta}{3}$
1	1.000				
0.5	1.250	+250	1.500		
0.25	1.514	+264	1.778	$+\frac{278}{3}$	$\boxed{1.871}$

Improvements of Euler's Method

Heun's Method

$O(h^2)$

- predictor-corrector approach
- average of two derivatives for the interval

predictor

$$y^0_{i+1} = y_i + f(t_i, y_i)\,h \qquad \text{// use slope at } t_i$$

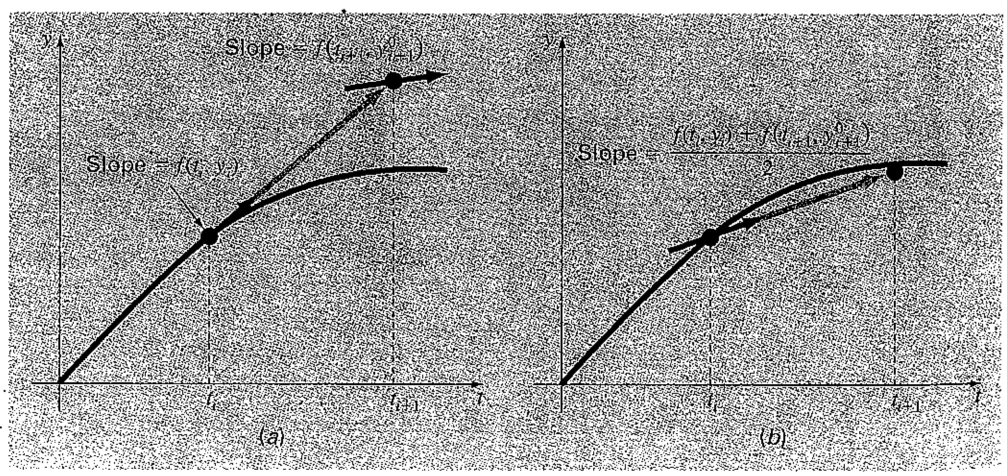
$$y'_{i+1} = f(t_{i+1}, y^0_{i+1}) \qquad \text{// slope at } t_{i+1}$$

$$\bar{y}_i = \frac{f(t_i, y_i) + f(t_{i+1}, y^0_{i+1})}{2} \qquad \text{// average}$$

corrector

$$y_{i+1} = y_i + \frac{f(t_i, y_i) + f(t_{i+1}, y^0_{i+1})}{2}\,h$$

FIGURE 20.4
Graphical depiction of Heun's method. (*a*) Predictor and (*b*) corrector.



The Midpoint Method

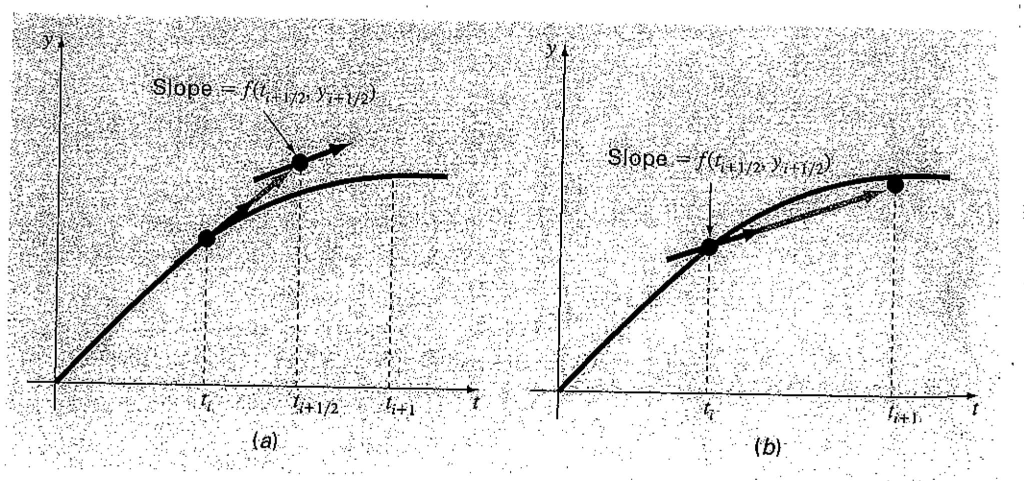
$O(h^2)$

$$y_{i+\frac{1}{2}} = y_i + f(t_i, y_i) \cdot \frac{h}{2}$$

$$y'_{i+\frac{1}{2}} = f(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

$$y_{i+1} = y_i + f(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}) \cdot h$$

FIGURE 20.6
Graphical depiction of ~~Heun's~~ Midpoint method. (*a*) Predictor and (*b*) corrector.



Example

$$\begin{cases} y' = 4 \cdot e^{0.8t} - 0.5y \\ y(0) = 2 \end{cases}$$

$[0, 4]$, $h = 1$

$y(4) = 75.33896$

Euler's	56.84931	(24.54% error)
Heun's (w/o)	83.33777	(10.62% error)
(with)	77.73510	(3.18% error)
Midpoint		

Runge-Kutta Method

CONTRIBUTE To this Entry

A method of numerically integrating ordinary differential equations by using a trial step at the midpoint of an interval to cancel out lower-order error terms. The second-order formula is

$$
\begin{aligned}
k_1 &= h f(x_n, y_n) && (1)\\
k_2 &= h f\left(x_n + \tfrac{1}{2} h, y_n + \tfrac{1}{2} k_1\right) && (2)\\
y_{n+1} &= y_n + k_2 + O\left(h^3\right), && (3)
\end{aligned}
$$

sometimes known as RK2, and the fourth-order formula is

$$
\begin{aligned}
k_1 &= h f(x_n, y_n) && (4)\\
k_2 &= h f\left(x_n + \tfrac{1}{2} h, y_n + \tfrac{1}{2} k_1\right) && (5)\\
k_3 &= h f\left(x_n + \tfrac{1}{2} h, y_n + \tfrac{1}{2} k_2\right) && (6)\\
k_4 &= h f(x_n + h, y_n + k_3) && (7)\\
y_{n+1} &= y_n + \tfrac{1}{6} k_1 + \tfrac{1}{3} k_2 + \tfrac{1}{3} k_3 + \tfrac{1}{6} k_4 + O\left(h^5\right) && (8)
\end{aligned}
$$

(Press *et al.* 1992), sometimes known as RK4. This method is reasonably simple and robust and is a good general candidate for numerical solution of differential equations when combined with an intelligent adaptive step-size routine.

Taylor series in two variables

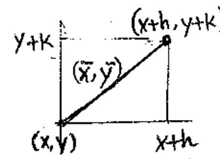
$$f(x+h,\, y+k) = \sum_{i=0}^{\infty} \frac{1}{i!} \left(h \cdot \frac{\partial}{\partial x} + k \cdot \frac{\partial}{\partial y}\right)^i f(x,y)$$

$$\left(h \cdot \frac{\partial}{\partial x} + k \cdot \frac{\partial}{\partial y}\right)^0 \cdot f(x,y) = f$$

$$\left(h \frac{\partial}{\partial x} + k \cdot \frac{\partial}{\partial y}\right)^1 \cdot f(x,y) = h \cdot \frac{\partial f}{\partial x} + k \cdot \frac{\partial f}{\partial y}$$

$$\left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^2 \cdot f(x,y) = h^2 \frac{\partial^2 f}{\partial x} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \cdot \frac{\partial^2 f}{\partial y^2}$$

$$\vdots$$



$$f(x+h, y+k) = \sum_{i=0}^{n-1} \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y}\right)^i f(x,y) + \underbrace{\frac{1}{n!} \left(h \cdot \frac{\partial}{\partial x} + k \cdot \frac{\partial}{\partial y}\right)^n \cdot f(\bar{x}, \bar{y})}_{\text{truncation error}}$$

Use $f_x \equiv \frac{\partial f}{\partial x}$, $f_t \equiv \frac{\partial f}{\partial t}$, $f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}$, $f_{xt} \equiv \frac{\partial^2 f}{\partial x \partial t}$ (Note. $f_{xt} = f_{tx}$)

$$\begin{aligned} f(x+h, y+k) = \; & f + (h \cdot f_x + k f_y) \\ & + \frac{1}{2!} (h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) \\ & + \frac{1}{3!} (h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}) \\ & + \cdots \end{aligned}$$

Note. $f(x+h, y) = f + h f_x + \frac{h^2}{2!} f_{xy} + \frac{h^3}{3!} f_{xxx} + \cdots$ // when $k=0$

$f(x, y+k) = f + k f_y + \frac{k^2}{2!} f_{yy} + \frac{k^3}{3!} f_{yyy} + \cdots$ // when $h=0$

Runge-Kutta Method of Order 2

Idea:

① Evaluate
$$\begin{cases} K_1 = h\cdot f(t,x) \\ K_2 = h\cdot f(t+\alpha h,\ x+\beta\cdot K_1) \end{cases}$$

② plug into

$$X(t+h) = X(t) + W_1 K_1 + W_2 K_2 \qquad // \text{ linear combination of } K_1 \text{ and } K_2$$

or
$$\boxed{X(t+h) = X(t) + \underline{W_1}\, h f(t,x) + \underline{W_2}\, h\cdot f(\underbrace{t}_{x} + \underbrace{\alpha h}_{h},\ \underbrace{x}_{y} + \underbrace{\beta h f(t,x)}_{K})} \quad \cdots (A)$$

③ Reproduce as many terms as possible in the Taylor series

$$\boxed{X(t+h) = X(t) + h\cdot \underbrace{X'(t)}_{=f} + \frac{1}{2!}h^2\cdot X''(t) + \frac{1}{3!}h^3 X'''(t) + \cdots} \quad \cdots (B)$$

We plan to match (A) and (B).

Note. when $\underline{W_1 = 1}$, $\underline{W_2 = 0}$, (A) and (B) agree up to 2nd term.

Apply 2-variable form of Taylor series to the final term of (A)

$$f(\underbrace{t}_{x} + \underbrace{\alpha h}_{h}, \underbrace{x}_{y} + \underbrace{\beta h f}_{K}) = f + \underbrace{\alpha h}_{h} f_t + \underbrace{\beta h f}_{K} f_x + \frac{1}{2}\left(\alpha h \frac{\partial}{\partial t} + \beta h f \frac{\partial}{\partial x}\right)^2 f(\bar{t},\bar{x})$$

Plug in (A),
$$\boxed{X(t+h) = X(t) + (W_1+W_2)hf + \alpha W_2 h^2 f_t + \beta W_2 h^2 f f_x + O(h^3)} \quad \cdots (A')$$

From
$$X(t+h) = X(t) + h\cdot \underbrace{X'(t)}_{=f} + \frac{1}{2}h^2 \underbrace{X''(t)} + \frac{1}{3!}h^3 X'''(t) + \cdots$$

Note. $X'' = \frac{dx'}{dt} = \frac{df(t,x)}{dt} = f_t + f\cdot f_x$

$$\therefore \boxed{X(t+h) = X(t) + hf + \frac{1}{2}h^2 f_t + \frac{1}{2}h^2 f f_x + O(h^3)} \quad \cdots (B')$$

$$W_1 + W_2 = 1,\quad \alpha W_2 = \frac{1}{2},\quad \beta W_2 = \frac{1}{2} \rightarrow \alpha = \beta = 1,\ W_1 = W_2 = \frac{1}{2}$$

Plug in (A), $\boxed{X(t+h) = X(t) + \frac{h}{2} f(t,x) + \frac{h}{2} f(t+h, x+h\cdot f(t,x))}$

or $\boxed{X(t+h) = X(t) + \frac{1}{2}(K_1+K_2)}$ where $K_1 = h\cdot f(t,x)$, $K_2 = h\cdot f(t+h, x+K_1)$

Classical Fourth-Order Runger-Kutta Method (RK4)

$$\boxed{y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

where

$$k_1 = f(t_i, y_i)$$

$$k_2 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} k_1\right)$$

$$k_3 = f\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} \cdot k_2\right)$$

$$k_4 = f(t_i + h, y_i + h \cdot k_3)$$

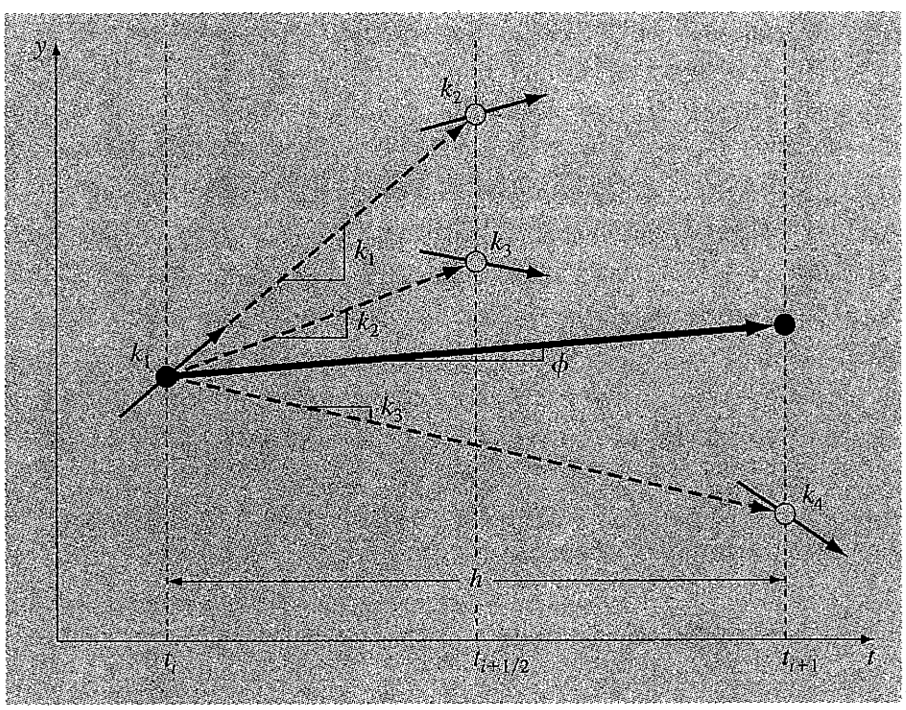


FIGURE 20.7
Graphical depiction of the slope estimates comprising the fourth-order RK method.

Ex 20.3

$$\begin{cases} y' = 4e^{0.8t} - 0.5y & \underline{t=[0,1]} \\ y(0) = 2 & h=1 \end{cases}$$

$$K_1 = f(0,2) = 4e^{0.8(0)} - 0.5(2) = 3$$

$$y(0.5) = 2 + 3(0.5) = 3.5$$

$$K_2 = f(0.5, 3.5) = 4e^{0.8(0.5)} - 0.5(3.5) = 4.217299$$

$$y(0.5) = 2 + 4.217299(0.5) = 4.108649$$

$$K_3 = f(0.5, 4.108649) = 4e^{0.8(0.5)} - 0.5(4.108649) = 3.912974$$

$$y(1.0) = 2 + 3.912974(1.0) = 5.912974$$

$$K_4 = f(1.0, 5.912974) = 4e^{0.8(1.0)} - 0.5(5.912974) = 5.945677$$

$$\phi = \frac{1}{6}[3 + 2(4.217299) + 2(3.912974) + 5.945677]$$

$$= 4.201037$$

$$y(1.0) = 2 + 4.201037(1.0) = 6.201037.$$

True value: 6.194631.

Error: 0.103 %

System of ODEs.

Ex. Bungee Jump

$$\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dv}{dt} = g - \dfrac{c_d}{m} v^2 \end{cases}$$

initial conditions

$$\begin{cases} x(0) = 0 \\ v(0) = 0 \end{cases}$$

[1] Euler's

$t=0$, $\dfrac{dx}{dt} = 0$

$$\frac{dv}{dt} = 9.81 - \frac{0.25}{68.1}(0)^2 = 9.81$$

$t=2$, $x = 0 + 0(2) = 0$

$$v = 0 + 9.81(2) = 19.62$$

$t=4$, $x = 0 + 19.62(2) = 39.24$

$$v = 19.62 + \left(9.81 - \frac{0.25}{68.1}\cdot(19.62)^2\right)2 = 36.41368$$

⋮

Euler's method.

t	x_{true}	v_{true}	x_{Euler}	v_{Euler}	$\varepsilon_t(x)$	$\varepsilon_t(v)$
0	0	0	0	0		
2	19.1663	18.7292	0	19.6200	100.00%	4.76%
4	71.9304	33.1118	39.2400	36.4137	45.45%	9.97%
6	147.9462	42.0762	112.0674	46.2983	24.25%	10.03%
8	237.5104	46.9575	204.6640	50.1802	13.83%	6.86%
10	334.1782	49.4214	305.0244	51.3123	8.72%	3.83%

[2] RK4

$$\frac{dx}{dt} = f_1(t,x,v) = v$$

$$\frac{dv}{dt} = f_2(t,x,v) = g - \frac{c_d}{m} v^2$$

$$K_{1,1} = f_1(0,0,0) = 0$$
$$K_{1,2} = f_2(0,0,0) = 9.81 - \frac{0.25}{68.1}(0)^2 = 9.81$$

$$x(1) = x(0) + K_{1,1}\frac{h}{2} = 0 + 0\cdot\frac{2}{2} = 0$$
$$v(1) = v(0) + K_{1,2}\cdot\frac{h}{2} = 0 + 9.81\,\frac{2}{2} = 9.81$$

$$K_{2,1} = f_1(1, 0, 9.81) = 9.81$$
$$K_{2,2} = f_2(1, 0, 9.81) = 9.4567$$

$$x(1) = x(0) + K_{2,1}\cdot\frac{h}{2} = 0 + 9.81\cdot\frac{2}{2} = 9.81$$
$$v(1) = v(0) + K_{22}\cdot\frac{h}{2} = 0 + 9.4567\cdot\frac{2}{2} = 9.4567$$

$$K_{3,1} = f_1(1, 9.81, 9.4567) = 9.4567$$
$$K_{3,2} = f_2(1, 9.81, 9.4567) = 9.4817$$

$$x(2) = x(0) + K_{3,1}\cdot h = 0 + 9.4567(2) = 18.9134$$
$$v(2) = v(0) + K_{3,2}\cdot h = 0 + 9.4817(2) = 18.9634$$

$$K_{4,1} = f_1(2, 18.9134, 18.9634) = 18.9634$$
$$K_{4,2} = f_2(2, 18.9134, 18.9634) = 8.4898$$

$$x(2) = 0 + \frac{1}{6}[0 + 2(9.81 + 9.4567) + 18.9634]\cdot 2 = \underline{19.1656}$$
$$v(2) = 0 + \frac{1}{6}[9.81 + 2(9.4567 + 9.4817) + 8.4896]\cdot 2 = \underline{18.7256}$$

t	x_{true}	v_{true}	x_{RK4}	v_{RK4}	$\varepsilon_t(x)$	$\varepsilon_t(v)$
0	0	0	0	0		
2	19.1663	18.7292	19.1656	18.7256	0.004%	0.019%
4	71.9304	33.1118	71.9311	33.0995	0.001%	0.037%
6	147.9462	42.0762	147.9521	42.0547	0.004%	0.051%
8	237.5104	46.9575	237.5104	46.9345	0.000%	0.049%
10	334.1782	49.4214	334.1626	49.4027	0.005%	0.038%

Higher-order differential equations

Rewrite as a system of first-order equations.

Ex

$$\begin{cases} x'' = x - y - (3x')^2 + (y')^3 + 6y'' + 2t \\ y''' = y'' - x' + e^x - t \\ x(1) = 2,\ x'(1) = -4,\ y(1) = -2,\ y'(1) = 7,\ y''(1) = 6 \end{cases}$$

①

Old Variable	New Variable	Inital value	Differential Equation
t	x_1	1	$x_1' = 1$
x	x_2	2	$x_2' = x_3$
x'	x_3	-4	$x_3' = x_2 - x_4 - 9x_3^2 + x_5^3 + 6x_6 + 2x_1$
y	x_4	-2	$x_4' = x_5$
y'	x_5	7	$x_5' = x_6$
y''	x_6	6	$x_6' = x_6 - x_3 + e^{x_2} - x_1$

$$X' = \begin{bmatrix} 1 \\ x^3 \\ x_2 - x_4 - 9x_3^2 + x_5^3 + 6x_6 + 2x_1 \\ x_5 \\ x_6 \\ x_6 - x_3 - e^{x_2} - x_1 \end{bmatrix}, \quad X(1) = \begin{bmatrix} 1 \\ 2 \\ -4 \\ -2 \\ 7 \\ 6 \end{bmatrix}$$

②

Apply RK4

coupled system

$$\frac{dx}{dt} = a(\underline{y}+b)x$$

$$\frac{dy}{dt} = c(\underline{x}+d)y$$

Ex $\begin{cases} x' = x - y + 2t - t^2 - t^3 \\ y' = x + y - 4t^2 + t^3 \end{cases}$ Note: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$

$\begin{cases} x(0) = 1 \\ y(0) = 0 \end{cases}$

Analytic solution: $\begin{cases} x(t) = e^t \cos(t) + t^2 \\ y(t) = e^t \sin(t) - t^3 \end{cases}$

Numerical solution:

(1) Taylor Series Method:

$$x(t+h) = x + hx' + \frac{h^2}{2}x'' + \frac{h^3}{6}x''' + \frac{h^4}{24}x^{iv} \cdots$$

$$y(t+h) = y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \frac{h^4}{24}y^{iv} \cdots$$

$$x' = x - y + 2t - t^2 - t^3 \qquad y' = x + y - 4t^2 + t^3$$

$$x'' = x' - y' + 2 - 2t - 3t^2 \qquad y'' = x' + y' - 8t + 3t^2$$

$$x''' = x'' - y'' - 2 - 6t \qquad y''' = x'' + y'' - 8 + 6t$$

$$x^{iv} = x''' - y''' - 6 \qquad y^{iv} = x''' + y''' + 6$$

$h = \frac{1}{n}$

$x = 1$; $y = 0$

$t = 0$

DO 10 K = 1, n

compute x', x'', x''', x^{iv}, y', y'', y''', y^{iv}

$X = x + hx' + \frac{h^2}{2}x'' + \frac{h^3}{6}x''' + \frac{h^4}{24}x^{iv}$

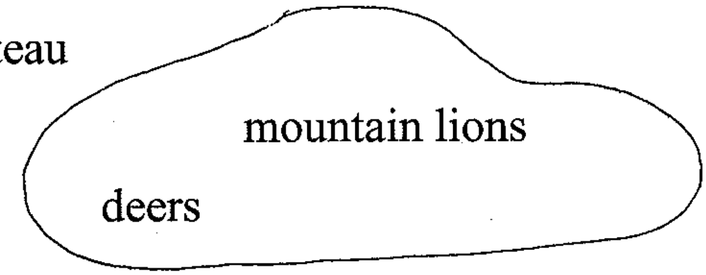
$Y = y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \frac{h^4}{24}y^{iv}$

$t = t + h$

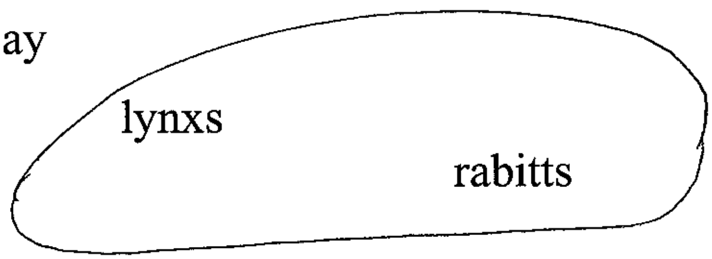
10 CONTINUE

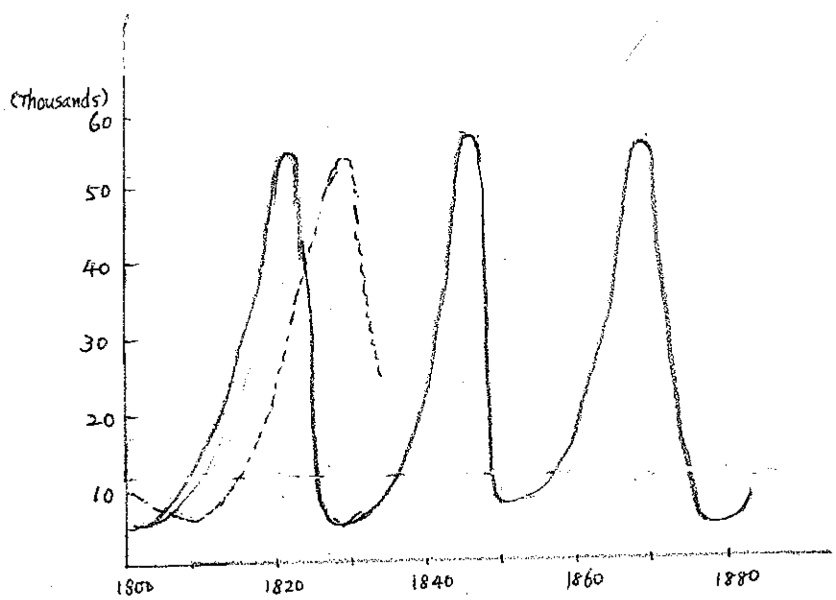
- "predator-prey model" in ecology

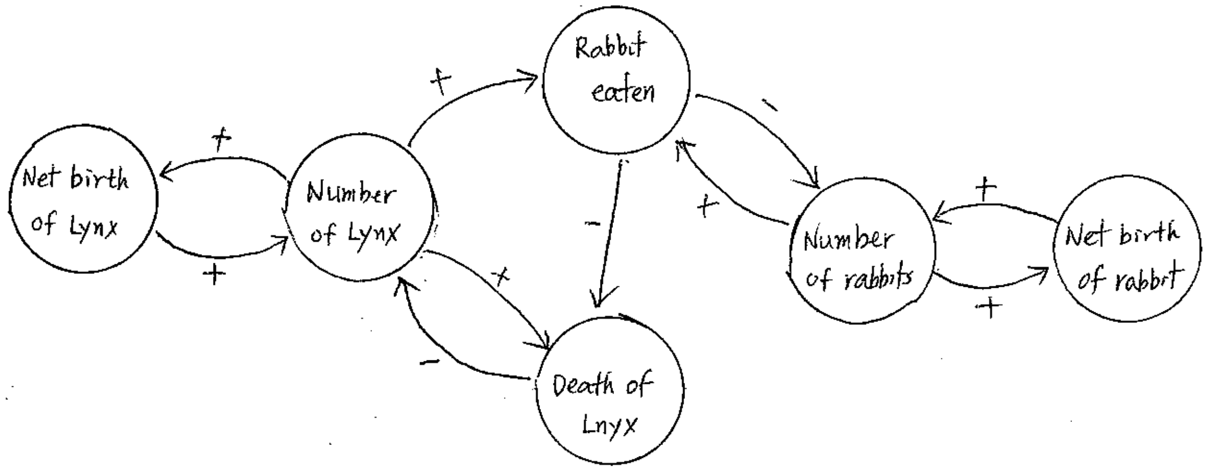
Kaibab Plateau



Hudson Bay






Net birth of Lynx
Number of Lynx
Rabbit eaten
Death of Lnyx
Number of rabbits
Net birth of rabbit
+
-

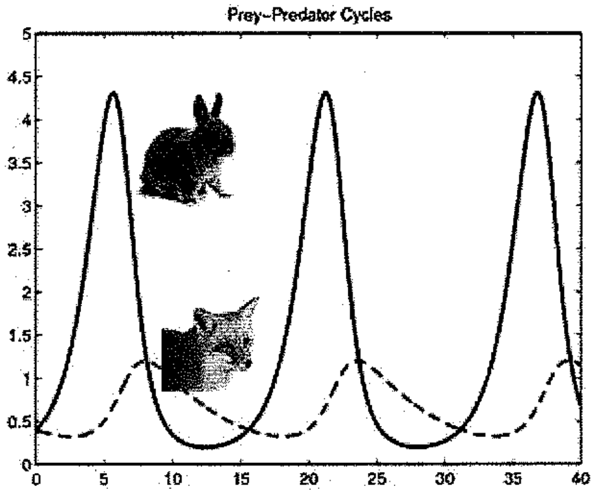


Figure 1: Periodic activity generated by the Predator-Prey model.

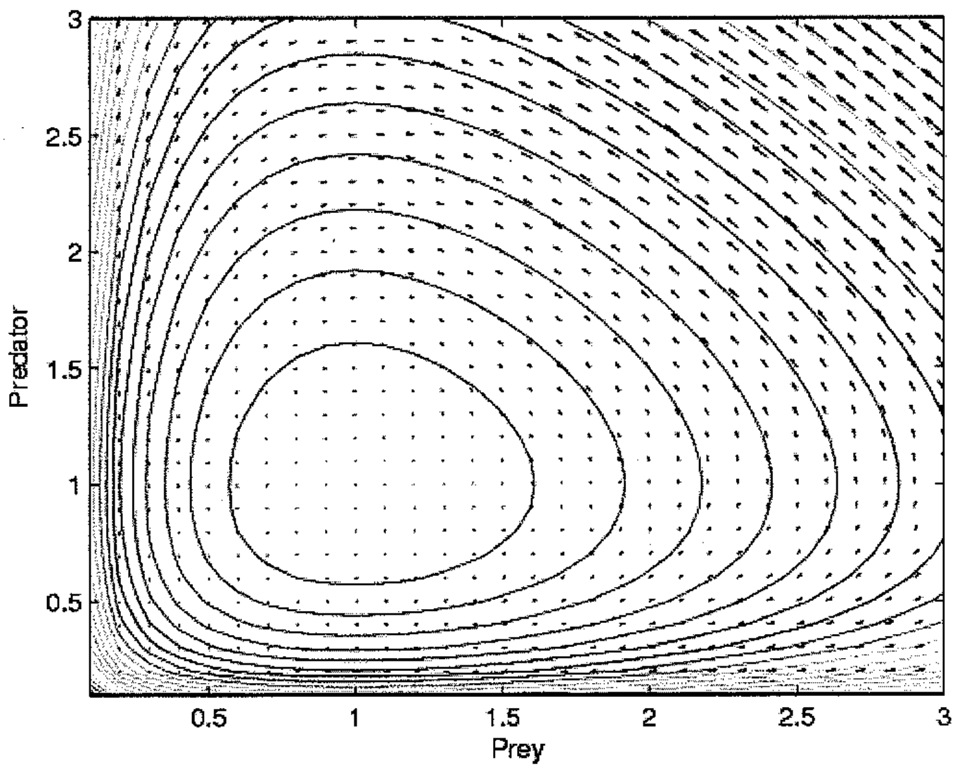


Figure 2: Prey-Predator dynamics as described by the level curves of a conserved quantity. The arrows describe the velocity and direction of solutions. In this simulation, the data are $d = r = b = d = 1$. There are equilibria at $x = 1, y = 1$ and at $x = 0, y = 0$

Lotka-Volterra Equation

- Alfred Lotka (1925) and Vito Volterra (1926)
- predator-prey model
- nonlinear differential equation.

$$\begin{cases} \dfrac{dx}{dt} = \underbrace{ax}_{\substack{\text{exponential}\\ \text{growth}}} - bxy & \cdots\cdots (1) \\ \dfrac{dy}{dt} = \underbrace{-cy}_{\substack{\text{exponential}\\ \text{decay}}} + dxy & \cdots\cdots (2) \end{cases}$$

x: number of prey

y: number of predator

t: time

$\dfrac{dx}{dt}$: growth rate of prey

$\dfrac{dy}{dt}$: growth rate of predator

a: prey growth rate

c: predator growth rate

b: effect of interaction on prey

d: effect of interaction on predator

Note

(1) changes in prey's number is given by its own growth minus the rate at which it is preyed upon.

(2) changes in predator's population as the growth of its own population, minus natural death.

Ex. Solve the Lotka-Volterra equation using the following parameter values and initial conditions.

$$\begin{cases} dx/dt = ax - bxy \\ dy/dt = -cy + dxy \end{cases}$$

$a = 1.2,\ b = 0.6,\ c = 0.8,\ d = 0.3$
$x(0) = 2,\ y(0) = 1$

Solution.

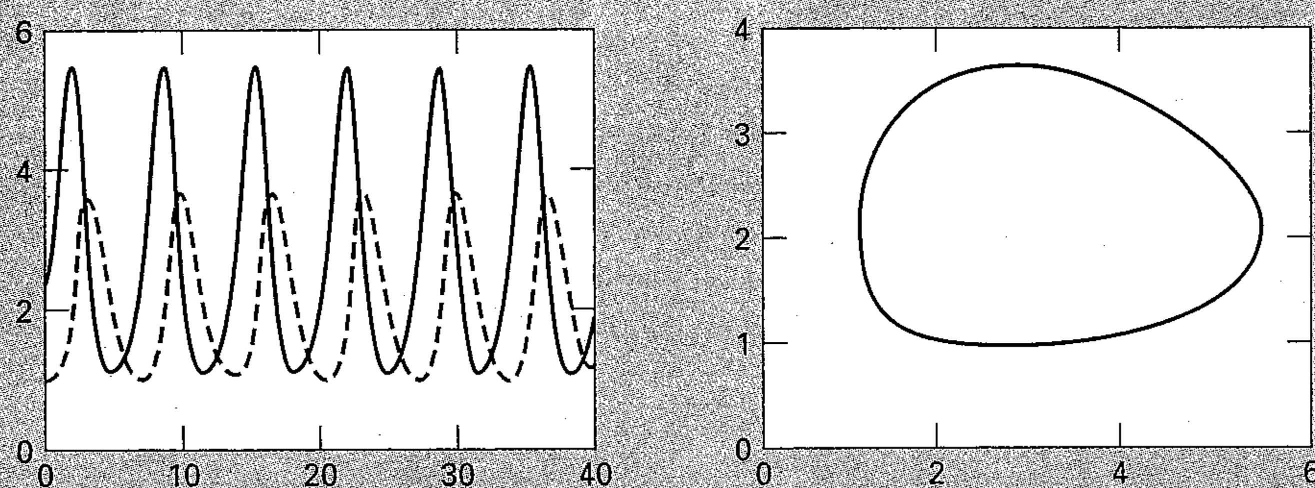
- Create M-file, pradprey.m

```
function  yp = pradprey(t,y)
yp = [1.2*y(1) – 0.6*y(1)*y(2); -0.8*y(2) + 0.3*y(1)*y(2)];
```

Note. $y(1) \equiv x$, $y(2) \equiv t$

-
```
>> tspan = [0 : 0.5 : 20]        // for every half year for 20 years
>> y0 = [2, 1];                  // #prey = 2, #predator = 1
>> [t, y] = ode45(@pradprey, tspan, y0)
>> plot(t,y)
```

t	y1(Prey)	y2(Predator)	t	y1(Prey)	y2(Predator)
0	2	1			
0.5	2.72	0.95	10.5	1.64	3.30
1	3.70	1.03	11	1.20	2.73
1.5	4.78	1.31	11.5	1.06	2.16
2	5.44	1.90	12	1.08	1.70
2.5	4.89	2.82	12.5	1.25	1.35
3	3.40	3.53	13	1.58	1.12
3.5	2.10	3.55	13.5	2.10	0.98
4	1.41	3.07	14	2.87	0.96
4.5	1.12	2.48	14.5	3.88	1.06
5	1.05	1.95	15	4.94	1.38
5.5	1.14	1.54	15.5	5.44	2.04
6	1.37	1.24	16	4.68	2.97
6.5	1.77	1.05	16.5	3.13	3.59
7	2.39	0.96	17	1.94	3.49
7.5	3.27	0.98	17.5	1.34	2.98
8	4.34	1.16	18	1.09	2.39
8.5	5.26	1.61	18.5	1.05	1.87
9	5.29	2.42	19	1.16	1.48
9.5	4.06	3.30	19.5	1.42	1.20
10	2.56	3.62	20	1.86	1.03



(c) RK4 time plot

(d) RK4 phase plane plot