Boundary-Value Problem

Ex. 1

$$\begin{cases} x''(t) = -x(t) \\ x(0) = 1, \ x(\frac{\pi}{2}) = -3 \end{cases}$$

General solution:

$$x(t) = c_1 \sin(t) + c_2 \cos(t)$$

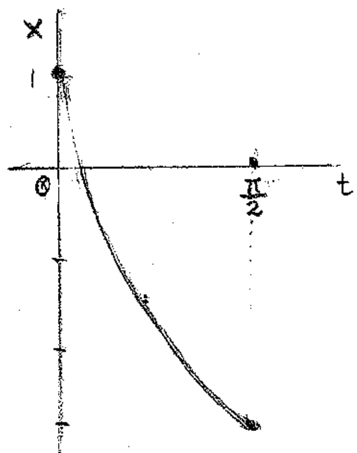
$$c_1 \overbrace{\sin(0)}^{0} + c_2 \overbrace{\cos(0)}^{1} = 1 \quad \rightarrow \quad c_2 = 1$$

$$c_1 \underbrace{\sin(\tfrac{\pi}{2})}_{1} + c_2 \underbrace{\cos(\tfrac{\pi}{2})}_{0} = -3 \quad \rightarrow \quad c_1 = -3$$

$$\therefore \boxed{x(t) = -3 \sin(t) + \cos(t)}$$

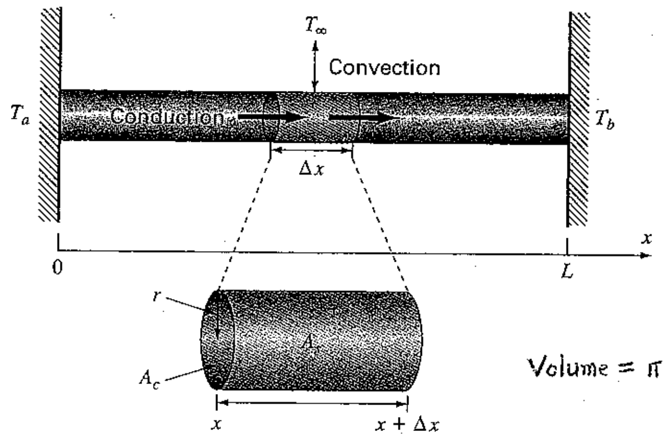
Check:

$$\left.\begin{aligned} x'(t) &= -3 \cos(t) - \sin(t) \\ x''(t) &= 3 \sin(t) - \cos(t) \\ &\parallel \\ -x(t) &= 3 \sin(t) - \cos(t) \end{aligned}\right\} \rightarrow \underline{x''(t) = -x(t)}$$



Motivation

Heat balance on heated rod



Volume $= \pi r^2 \cdot \Delta x$

$$q(x) \cdot Ac - q(x + \Delta x)\, Ac + h \cdot As\, (T_\infty - T) = 0 \tag{1}$$

where

$q(x)$:	flux into the element due to conduction
$q(x+\Delta x)$:	flux out of the element due to conduction
Ac:	cross-section area (πr^2)
r:	radius
h:	convection heat transfer coefficient
As:	element's surface area $(2\pi r \cdot \Delta x)$
T_∞:	air temperature
T:	rod temperature in K

Divide (1) by element's volume $(\pi r^2 \Delta x)$

$$\frac{q(x) - q(x+\Delta x)}{\Delta x} + \frac{2h}{r}(T_\infty - T) = 0 \tag{2}$$

Taking the limit $\Delta x \to 0$

$$-\frac{dq}{dx} + \frac{2h}{r}(T_\infty - T) = 0 \tag{3}$$

$$q = -k \frac{dT}{dx} \qquad \text{(From Fourier's Law)} \qquad (4)$$

where k is thermal conductivity

Differentiate,

$$\frac{dq}{dx} = -K \cdot \frac{d^2T}{dx}$$

Put into (3),

$$K \cdot \frac{d^2T}{dx^2} + \frac{2h}{r}(T_\infty - T) = 0$$

Divide by K,

$$\frac{d^2T}{dx^2} + \overbrace{\frac{2h}{r \cdot K}}^{h'}(T_\infty - T) = 0$$

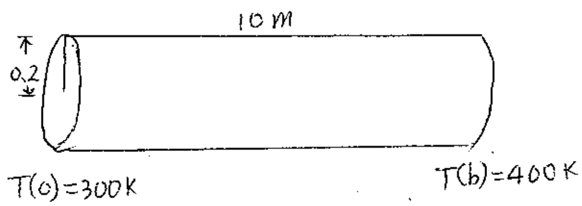
// $h' \equiv$ bulk heat-transfer parameter.

2nd-Order ODE

boundary conditions

$$T(0) = T_a$$
$$T(L) = T_b$$

Analytic Solution



$h' = 0.05$, $k = 200$, $T_\infty = 200$

Solution

$$\frac{d^2T}{dx^2} - h'T = -h'T_\infty$$

- Homogeneous solution:

$$\frac{d^2T}{dx^2} - h'T = 0$$

Assume $T = e^{\lambda x}$

$$\lambda^2 e^{\lambda x} - h' e^{\lambda x} = 0$$

$$(\lambda^2 - h')\, e^{\lambda x} = 0 \quad \Rightarrow \lambda = \pm\sqrt{h'}$$

$$T = A \cdot e^{\lambda x} + B\, e^{-\lambda x}$$

- Particular solution $\Rightarrow T = T_\infty$

- General solution:

$$T = T_\infty + A\, e^{\lambda x} + B\, e^{-\lambda x}$$

From boundary conditions,

$$\begin{cases} T_a = T_\infty + A + B \\ T_b = T_\infty + A\, e^{\lambda L} + B\, e^{-\lambda L} \end{cases}$$

$$A = \frac{(T_a - T_\infty)e^{-\lambda L} - (T_b - T_\infty)}{e^{-\lambda L} - e^{\lambda L}} = 20.4671$$

$$B = \frac{(T_b - T_\infty) - (T_a - T_\infty)\, e^{\lambda L}}{e^{-\lambda L} - e^{\lambda L}} = 79.5329.$$

$$\therefore T = 200 + 20.4671\, e^{\sqrt{0.05}\,x} + 79.5329\, e^{-\sqrt{0.05}\,x}$$

Shooting Method

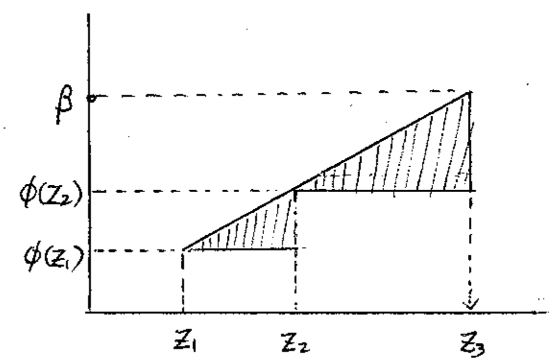
$$\begin{cases} x''(t) = f(t, x(t), x'(t)) \\ x(a) = \alpha, \quad x(b) = \beta \end{cases}$$

Steps:

(1) Guess $x''(a) \equiv z_1$ and solve the initial-value problem $\Rightarrow \phi(z_1)$

(2) Guess $x'(a) \equiv z_2$ and solve the initial-value problem $\Rightarrow \phi(z_2)$

(3) Interpolation



$$\frac{z_3 - z_2}{\beta - \phi(z_2)} = \frac{z_2 - z_1}{\phi(z_2) - \phi(z_1)} \Rightarrow z_3 = z_2 + \frac{[\beta - \phi(z_2)](z_2 - z_1)}{\phi(z_2) - \phi(z_1)}$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow z_3 \qquad \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} \longrightarrow z_4 \qquad \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \longrightarrow z_5 \quad \cdots$$

Modification/Refinement

Cubic interpolating polynomial

In general,

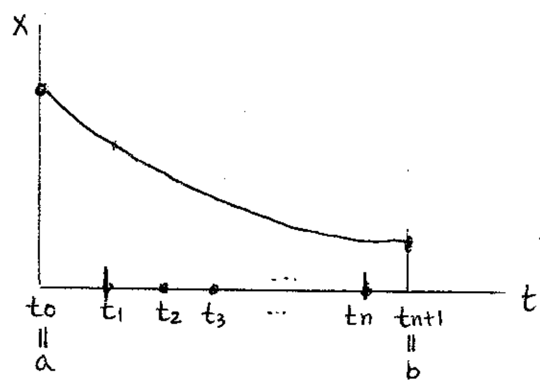
$$\boxed{z_{n+1} = z_n + \frac{[\beta - \phi(z_n)](z_n - z_{n-1})}{\phi(z_n) - \phi(z_{n-1})} \qquad n \geq 2}$$

Monitor $\phi(z_{n+1}) - \beta$

Finite Difference Approximation

$$\begin{cases} x'' = f(t, x, x') \\ x(a) = \alpha, \quad x(b) = \beta \end{cases}$$

$$t_i = a + ih, \quad h = \frac{b-a}{n+1}$$



Approximate the derivatives with central difference:

$$x'(t) \approx \frac{1}{2h}[x(t+h) - x(t-h)]$$

$$x''(t) \approx \frac{1}{h^2}[x(t+h) - 2x(t) + x(t-h)]$$

Solve

$$\begin{cases} x_0 = \alpha \\ \frac{1}{h^2}(x_{i-1} - 2x_i + x_{i+1}) = f\left(t_i, x_i, \frac{1}{2h}(x_{i+1} - x_{i-1})\right), \ 1 \le i \le n \\ x_{n+1} = \beta \end{cases} \quad \cdots\cdots (A)$$

When the system (A) is linear:

$$f(t,x,x') = u(t) + v(t)x + w(t)x'$$

$$\frac{1}{h^2}(x_{i-1} - 2x_i + x_{i+1}) = u(t_i) + v(t_i)x_i + w(t_i)\left[\frac{1}{2h}(x_{i+1} - x_{i-1})\right]$$

Rearrange,

$$\underbrace{-(1+\tfrac{h}{2}w_i)}_{a_i} x_{i-1} + \underbrace{(2+h^2 v_i)}_{d_i} x_i \underbrace{-(1-\tfrac{h}{2}w_i)}_{c_i} x_{i+1} = \underbrace{-h^2 u_i}_{b_i},$$

wher $h = \frac{b-a}{(n+1)}$, $t_i = a + ih$ for $1 \le i \le n$.

$$a_i x_{i-1} + d_i x_i + c_i x_{i+1} = b_i \qquad 1 \le i \le n$$

$$\begin{cases} d_1 x_1 + c_1 x_2 = b_1 - a_1\alpha & i=1 \\ a_i x_{i-1} + d_i x_i + c_i x_{i+1} = b_i & 2 \le i \le n-1 \\ a_n x_{n-1} + d_n x_n = b_n - c_n\beta & i=n \end{cases}$$

or

$$\begin{bmatrix} d_1 & c_1 & & & & 0 \\ a_2 & d_2 & c_2 & & & \\ & a_3 & d_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & d_{n-1} & c_{n-1} \\ 0 & & & & a_n & d_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 - a_1\alpha \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n - c_n\beta \end{bmatrix} \quad \cdots\cdots (B)$$

Solve (B).

Note. Triangular system.