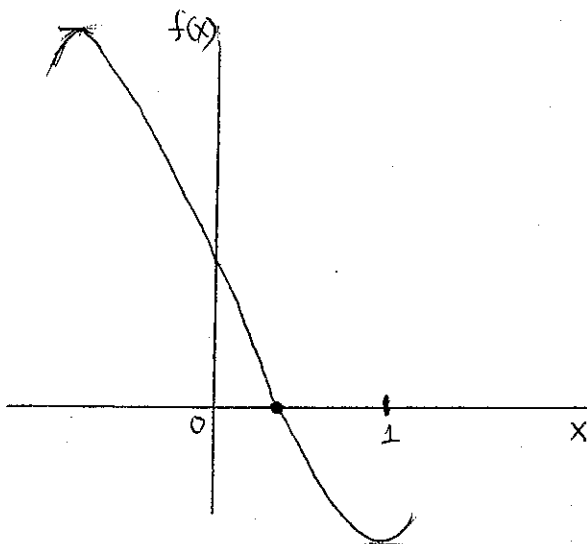


Chapter 5,6 Finding Roots

[1] Graphical Method

$$f(x) = x^3 - 3x + 1 \quad [0, 1]$$

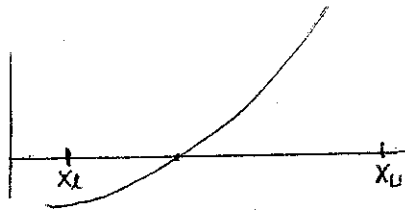
$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ &= 3(x+1)(x-1) \end{aligned}$$



Note. imprecise
rough estimate (near 3.3)

Incremental Search Method

Idea:



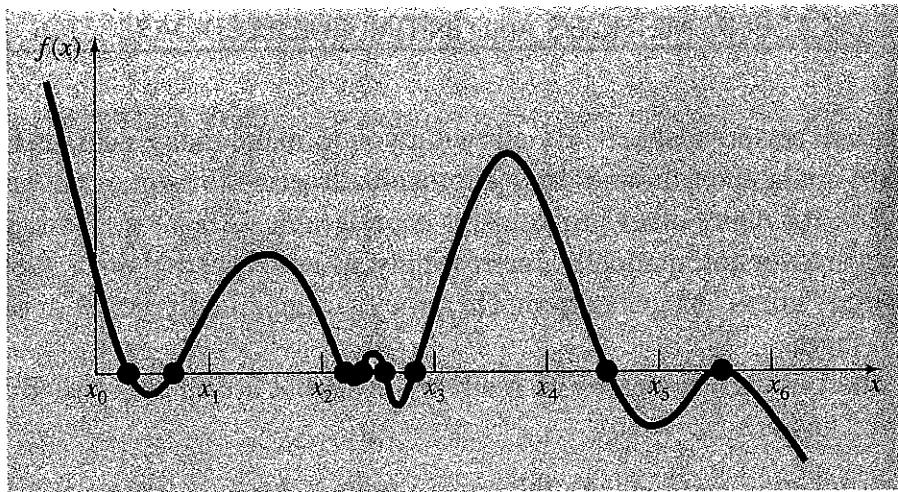
$$f(x_l) \cdot f(x_u) < 0$$

Find the interval(s) where the function changes sign.

Problem: How to choose increment length?

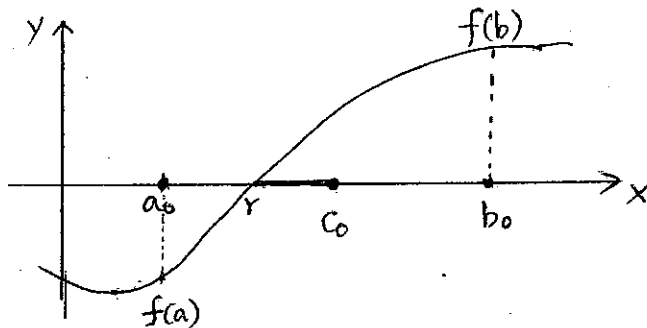
If too small, large amount of computation needed.

If too large, may miss root(s)



- | | |
|---------------|---------------------|
| $(x_0 - x_1)$ | miss multiple roots |
| $(x_2 - x_3)$ | miss multiple roots |
| $(x_4 - x_5)$ | o.k. |
| $(x_5 - x_6)$ | miss always |

[2] Bisection Method

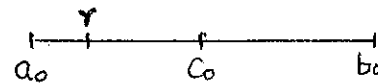


$$f(a) \cdot f(b) < 0$$

Note: Intermediate-value theorem

Convergence Analysis

$$r: [a_0, b_0]$$



$$c_0 = \frac{(a_0 + b_0)}{2}$$

$$r - c_0 \leq \frac{b_0 - a_0}{2}$$

$$r - c_1 \leq \frac{b_1 - a_1}{2}$$

...

$$r - c_n \leq \frac{b_n - a_n}{2} = \frac{b_{n-1} - a_{n-1}}{2^2} = \dots = \frac{b_1 - a_1}{2^n} = \frac{b_0 - a_0}{2^{n+1}}$$

$$\boxed{r - c_n \leq \frac{b_0 - a_0}{2^{n+1}}} \quad - \text{ error bound after } n \text{ steps}$$

1. Example.

$$f(x) = x^3 - 3x + 1 \quad [0, 1]$$

Iteration	X_l	X_u	X_r	Diagram
1	0	1	0.5	
2	0	0.5	0.25	
3	0.25	0.5	0.375	
4	0.25	0.375	0.3125	
5	0.3125	0.375	0.34375	
6	0.34375	0.375	0.359375	
7	0.34375	0.359375	0.3515625	
8	0.34375	0.3515625	0.3476562	
9	0.34375	0.3476562	0.3457031	
10	0.3457031	0.3476562	0.3466936	
11	0.3466936	0.3476562	0.3471749	

How many iterations are needed to achieve error bound ϵ ?

$$\text{If } |r - C_n| < \epsilon \text{ then } \frac{b-a}{2^{n+1}} < \epsilon$$

Take logarithm,

$$\log(b-a) - \log 2^{n+1} < \log \epsilon$$

$$\log(b-a) - (n+1) \log 2 < \log \epsilon$$

$$n \log 2 > \log(b-a) - \log \epsilon - \log 2$$

$$n > \frac{\log(b-a) - \log 2\epsilon}{\log 2}$$

Example.

Find n when $a = 0.1$, $b = 1.0$, and $\epsilon = 1/2 \cdot 10^{-8}$

Solution

$$n > \frac{\log(1.0 - 0.1) - \log 2 \cdot \frac{1}{2} \cdot 10^{-8}}{\log 2} = 26.42$$

\therefore 27 iterations are needed to achieve the error bound.

procedure BISECT (f, a₀, b₀, ε, root)

// f(a₀) < 0 and f(b₀) > 0 //

k = 0

found = false

while (not found) and (b_k - a_k) > ε do

c_k = 1/2 (a_k + b_k)

case of f(c_k)

0: found = true

-: (a_{k+1}, b_{k+1}) = (c_k, b_k)

+: (a_{k+1}, b_{k+1}) = (a_k, c_k)

endcase

k = k + 1

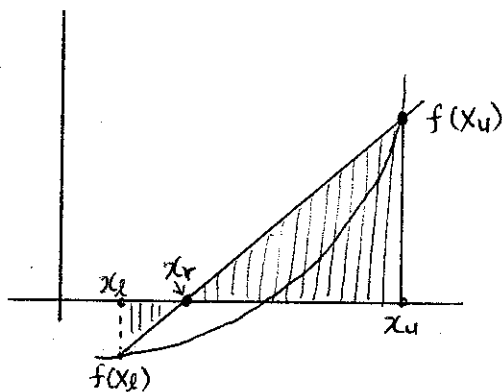
endwhile

if found then root = c_{k-1}

else root = (b_k + a_k) / 2

endif

Note. False Position (Linear Interpolation Method)



$$\text{Use } x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

Not recommended!

Fixed-Point Iteration Method

$$x = g(x)$$

$$x_{i+1} = g(x_i)$$

Ex. 6.1 $f(x) = e^{-x} - x$ (root = 0.56714329)

$$x_{i+1} = e^{-x_i} \quad (e = 2.71828182\dots)$$

x_i	error	$ \epsilon_t \%$
$x_0 = 0$	0.567	100
$x = e^{-0} = 1$	0.433	76.37
$x = e^{-1} = 0.3679$	0.1991	35.11
$x = e^{-0.3679} = 0.6922$	0.1252	22.08
$x = e^{-0.6922} = 0.5005$	0.0667	11.74
\vdots	\vdots	\vdots
$x = \dots = 0.5649$	0.0022	0.395

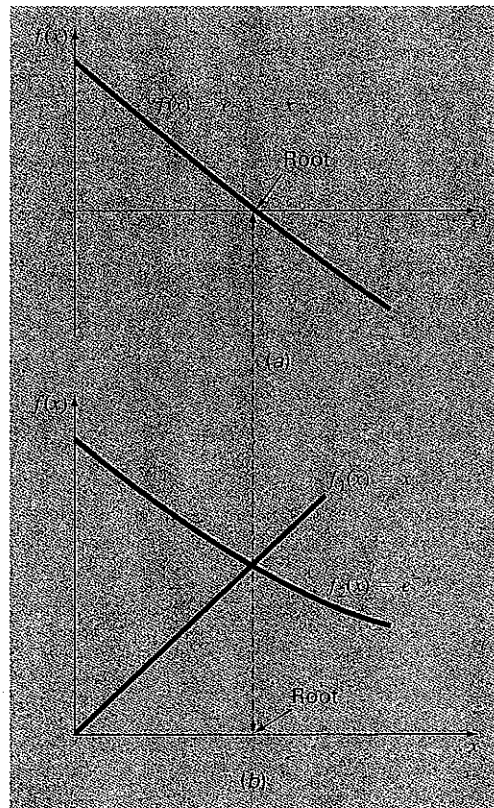


FIGURE 6.2

Two alternative graphical methods for determining the root of $f(x) = e^{-x} - x$: (a) point where it crosses the x axis; (b) root at the intersection of the component fun

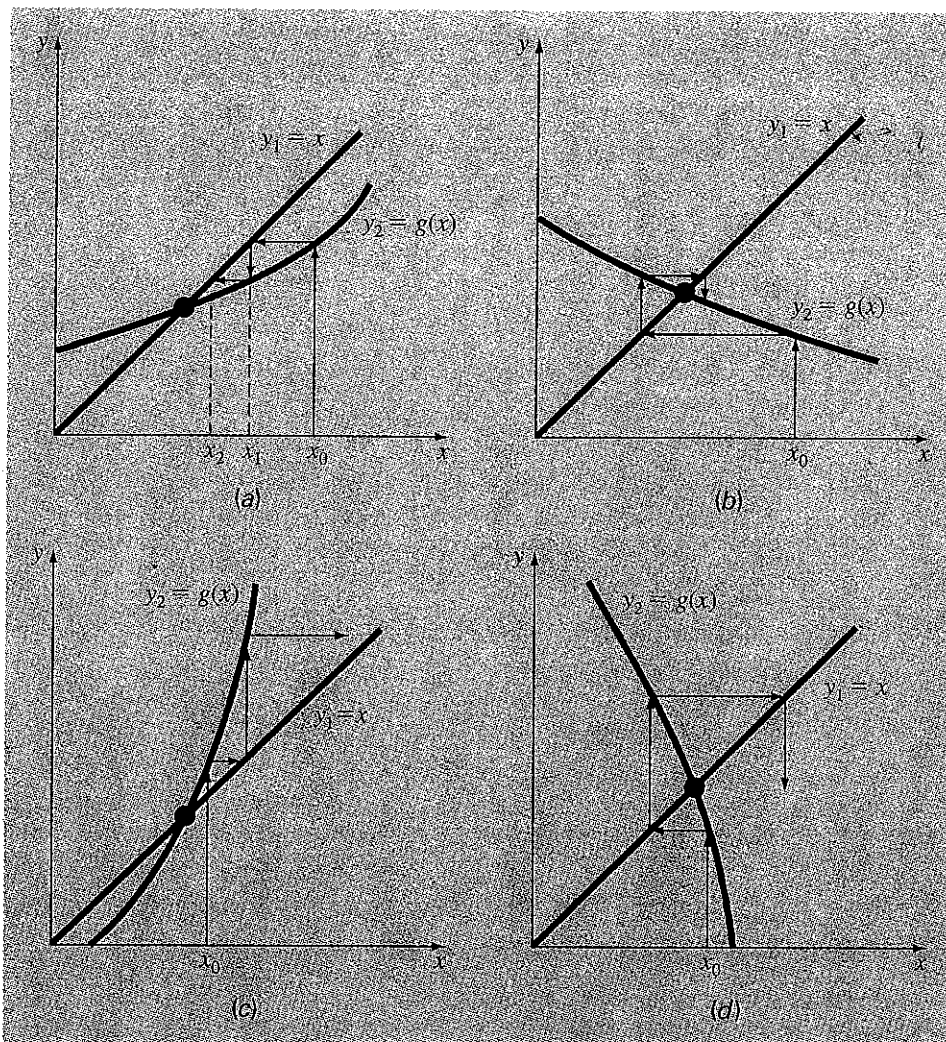
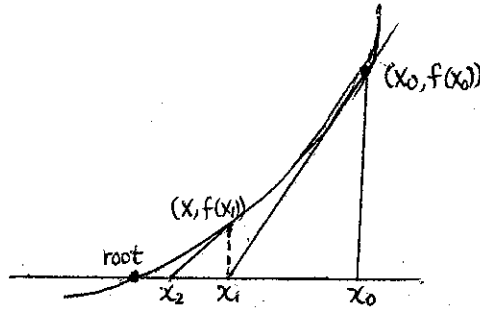


FIGURE 6.3

Graphical depiction of (a) and (b) convergence and (c) and (d) divergence of simple fixed-point iteration. Graphs (a) and (c) are called monotone patterns whereas (b) and (d) are called oscillating or spiral patterns. Note that convergence occurs when $|g'(x)| < 1$.

[3] Newton's Method (Newton-Raphson's Method)

(1) Idea



$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

⋮

Note, $f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$

$$x_{n+1} = x_n + h_n, \quad h_n = -\frac{f(x_n)}{f'(x_n)}$$

Iterative method

(2) We want to find h in $f(x_0 + h) = 0$ from Taylor series.

$$f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \dots = 0 \rightarrow h = -\frac{f(x_0)}{f'(x_0)}$$

$$\begin{aligned} \therefore x_1 &= x_0 + h \\ &= x_0 - \frac{f(x_0)}{f'(x_0)} \end{aligned}$$

Example

$$f(x) = \sin x - \left(\frac{x}{2}\right)^2$$

$$f'(x) = \cos x - x/2$$

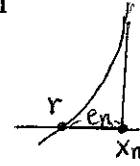
Let $x_0 = 1.5$

n	x	f(x _n)	f'(x _n)	h _n
0	1.5	0.434995	-0.67926	0.64039
1	2.14039	-0.303197	-1.60948	-0.18838
2	1.95201	-0.024372	-1.34805	-0.01808
3	1.93393	-0.000233	-1.32217	-0.00018
4	1.93375	0.000005	-1.32191	< 1/2 · 10 ⁻⁵

Convergence Analysis

Assume $f(x)$ has two continuous derivatives, f' and f'' , and the root r is a simple root. Then $f'(r) \neq 0$ and $f'(x) \neq 0$ for all x in a certain neighborhood of the root r .

Let e_n be error in the estimate x_n i.e. $e_n = x_n - r$



Expand f in a Taylor series about x_n

$$0 = f(r) = f(x_n - e_n) = f(x_n) - (x_n - r) f'(x_n) + \frac{(x_n - r)^2}{2!} f''(\xi), \quad \xi \in \text{int}(x_n, r)$$

Divide by $f'(x_n)$,

$$\frac{f(x_n)}{f'(x_n)} - x_n + r = \frac{\frac{1}{2}(r - x_n)^2 f''(\xi)}{f'(x_n)}$$

$$r - \underbrace{\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}_{x_{n+1}} = \frac{\frac{1}{2}(r - x_n)^2 f''(\xi)}{f'(x_n)}$$

$$\underbrace{e_{n+1}}_{e_{n+1}} = \frac{1}{2} e_n^2 \cdot \frac{f''(\xi)}{f'(\xi)}$$

$$\text{As } x_n \rightarrow r, \quad e_{n+1} \rightarrow \frac{1}{2} \frac{f''(r)}{f'(r)} e_n^2 = c \cdot e_n^2$$

I.e. $|x_{n+1} - r| \leq c |x_n - r|^2$: quadratic convergent

Every iteration doubles significant digits.

Q. Why Newton's method converge faster than Bisection method?

A. Newton's method uses more information than Bisection method.

() ()

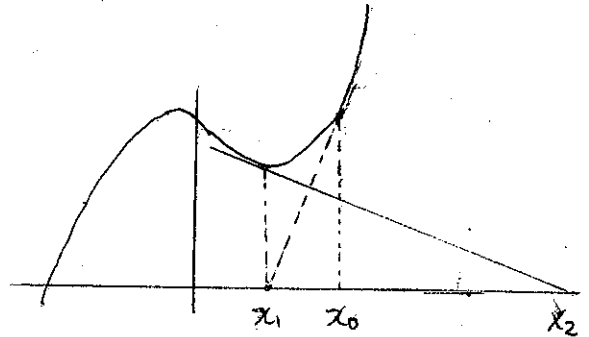
Ex. $f(x) = x^3 - x + 1, \quad x_c = 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad f'(x) = 3x^2 - 1, \quad f'(1) = 2$$

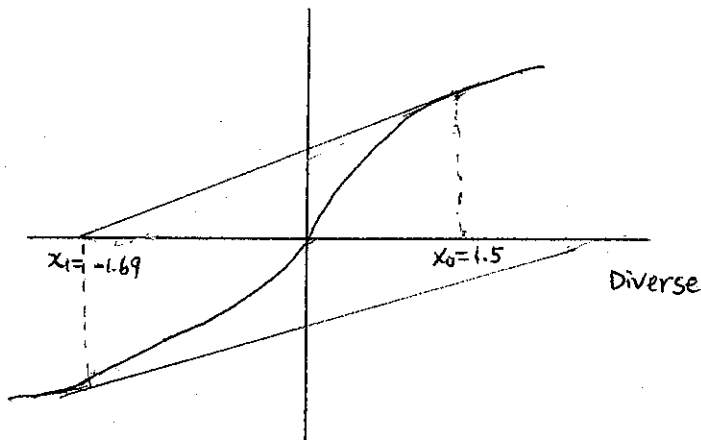
$$\therefore x_1 = 1/2$$

$$f(1/2) = 5/8, \quad f'(1/2) = -1/4$$

$$\therefore x_2 = 1/2 - \frac{5/8}{-1/4} = \frac{1}{2} + \frac{5}{2} = 3$$



Ex. $f(x) = \text{Arctan } x$



Why diverse? Initial point, x_0 , was not good.

Suggestion. Use graph method or bisection method to find a good initial point. Then switch to Newton's method for faster convergence.

Note. (1) Newton's method can be used for nonlinear equations.

(2) Algebraic solutions for up to 4th degree polynomials.

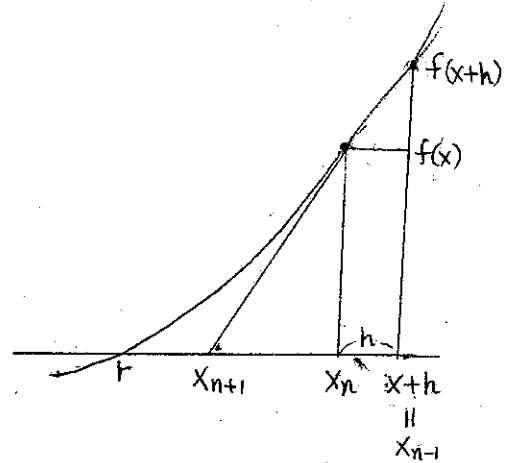
[4] Secant Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \dots \quad (A)$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

for small h ,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$



Let $x = x_n$, $h = x_{n-1} - x_n$

$$\text{Then } f'(x) \approx \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

From (A),

$$x_{n+1} = x_n - f(x_n) \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \right)$$

Note. needs 2 starting points.

Modified Secant Method

$$f'(x_i) \approx \frac{f(x_i + \delta x_i) - f(x_i)}{\delta x_i}$$

δx_i : small perturbation

Plug in into Newton's

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{\delta x_i \cdot f(x_i)}{f(x_i + \delta x_i) - f(x_i)}$$

$\delta x_i = x_i \cdot \text{perturbation fraction}$

EXAMPLE 6.5 Modified Secant Method

Problem Statement. Use the modified secant method to determine the mass of the bungee jumper with a drag coefficient of 0.25 kg/m to have a velocity of 36 m/s after 4 s of free fall. Note: The acceleration of gravity is 9.81 m/s². Use an initial guess of 50 kg and a value of 10^{-6} for the perturbation fraction.

Solution. Inserting the parameters into Eq. (6.9) yields

First iteration:

$$\begin{aligned} x_0 &= 50 & f(x_0) &= -4.57938708 \\ x_0 + \delta x_0 &= 50.00005 & f(x_0 + \delta x_0) &= -4.579381118 \\ x_1 &= 50 - \frac{\overset{\delta x_0}{10^{-6}(50)}(-4.57938708)}{-4.579381118 - (-4.57938708)} \\ &= 88.39931 (|e_r| = 38.1\%; |e_a| = 43.4\%) \end{aligned}$$

Second iteration:

$$\begin{aligned} x_1 &= 88.39931 & f(x_1) &= -1.69220771 \\ x_1 + \delta x_1 &= 88.39940 & f(x_1 + \delta x_1) &= -1.692203516 \\ x_2 &= 88.39931 - \frac{\overset{\delta x_1}{10^{-6}(88.39931)}(-1.69220771)}{-1.692203516 - (-1.69220771)} \\ &= 124.08970 (|e_r| = 13.1\%; |e_a| = 28.76\%) \end{aligned}$$

The calculation can be continued to yield

i	x_i	$ e_r , \%$	$ e_a , \%$
0	50.0000	64.971	
1	88.3993	38.069	43.438
2	124.0897	13.064	28.762
3	140.5417	1.538	11.706
4	142.7072	0.021	1.517
5	142.7376	4.1×10^{-6}	0.021
6	142.7376	3.4×10^{-12}	4.1×10^{-6}