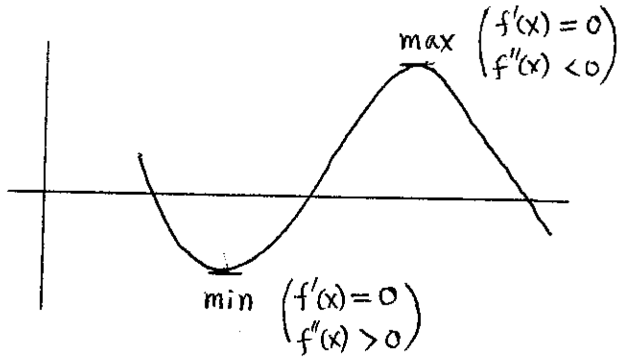
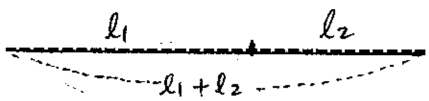
Chapter 7. Optimization



- Golden ratio (1 : 1.618)

(1)



$$\frac{\ell_1 + \ell_2}{\ell_1} = \frac{\ell_1}{\ell_2} = \phi$$

$$1 + \frac{1}{\phi} = \phi$$

$$\phi + 1 = \phi^2$$

$$\therefore \underline{\phi^2 - \phi - 1 = 0}$$

$$\phi = \frac{1 \pm \sqrt{1 - (-4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\phi \fallingdotseq 1.618$$

(2) Fibonacci Sequence

1170 ~ 1250

1 1 2 3 5 8 13 21 34 55 89 …

ratio 1 2 1.5 1.66 1.6 1.625 1.6254 1.619 1.61765 1.61818 …

Recurrence relation (Difference equation)

$$
\begin{cases}
F(1) = 1 & n = 1 \\
F(2) = 1 & n = 2 \\
F(n) = F(n-1) + F(n-2) & n > 2
\end{cases}
$$

$$F(n) - F(n-1) - F(n-2) = 0$$

Let $F(n) = \alpha^n$

$$\alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0$$

$$\alpha^{n-2}(\alpha^2 - \alpha - 1) = 0$$

$$\alpha^2 - \alpha - 1 = 0$$

$$\alpha = \frac{1+\sqrt{5}}{2}$$

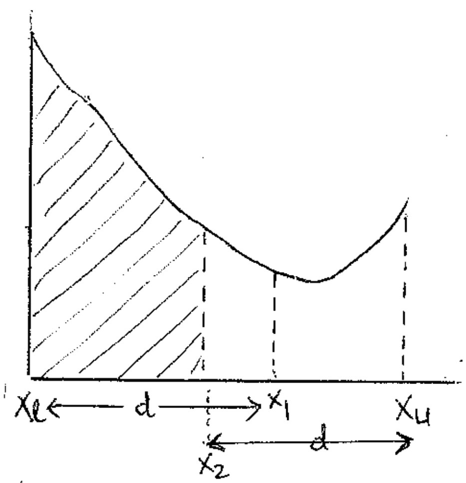
$$\fallingdotseq 1.618$$

Golden-Section Search

Choose $\begin{cases} x_1 = x_\ell + d \\ x_2 = x_u - d \end{cases}$

where $d = \underbrace{(\phi - 1)}_{0.618} (x_u - x_\ell)$

Case 1. $f(x_1) < f(x_2)$

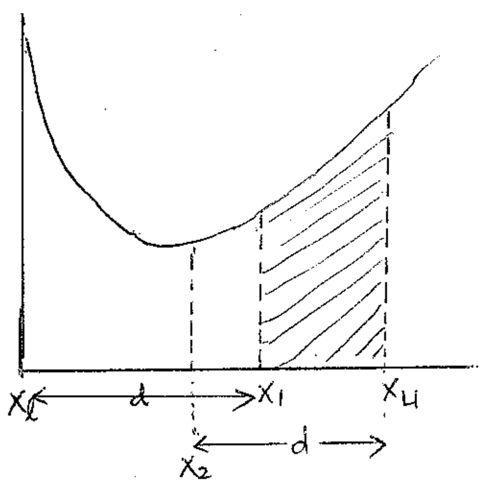


old interval (x_ℓ, x_u)

$\downarrow \quad \downarrow$

new interval (x_2, x_u)

Case 2. $f(x_1) > f(x_2)$



old interval (x_ℓ, x_u)

$\downarrow \quad \downarrow$

new interval (x_ℓ, x_1)

EXAMPLE 7.2 Golden-Section Search

Problem Statement. Use the golden-section search to find the minimum of

$$f(x) = \frac{x^2}{10} - 2\sin x$$

within the interval from $x_l = 0$ to $x_u = 4$.

Solution. First, the golden ratio is used to create the two interior points:

$$d = 0.61803(4 - 0) = 2.4721$$
$$x_1 = 0 + 2.4721 = 2.4721$$
$$x_2 = 4 - 2.4721 = 1.5279$$

The function can be evaluated at the interior points:

$$f(x_2) = \frac{1.5279^2}{10} - 2\sin(1.5279) = -1.7647$$

$$f(x_1) = \frac{2.4721^2}{10} - 2\sin(2.4721) = -0.6300$$

Because $f(x_2) < f(x_1)$, our best estimate of the minimum at this point is that it is located at $x = 1.5279$ with a value of $f(x) = -1.7647$. In addition, we also know that the minimum is in the interval defined by x_l, x_2, and x_1. Thus, for the next iteration, the lower bound remains $x_l = 0$, and x_1 becomes the upper bound, that is, $x_u = 2.4721$. In addition, the former x_2 value becomes the new x_1, that is, $x_1 = 1.5279$. In addition, we do not have to recalculate $f(x_1)$, it was determined on the previous iteration as $f(1.5279) = -1.7647$.

All that remains is to use Eqs. (7.8) and (7.7) to compute the new value of d and x_2:

$$d = 0.61803(2.4721 - 0) = 1.5279$$
$$x_2 = 2.4721 - 1.5279 = 0.9443$$

The function evaluation at x_2 is $f(0.9943) = -1.5310$. Since this value is less than the function value at x_1, the minimum is $f(1.5279) = -1.7647$, and it is in the interval prescribed by x_2, x_1, and x_u. The process can be repeated, with the results tabulated here:

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d
1	0	0	1.5279	−1.7647	2.4721	−0.6300	4.0000	3.1136	2.4721
2	0	0	0.9443	−1.5310	1.5279	−1.7647	2.4721	−0.6300	1.5279
3	0.9443	−1.5310	1.5279	−1.7647	1.8885	−1.5432	2.4721	−0.6300	0.9443
4	0.9443	−1.5310	1.3050	−1.7595	1.5279	−1.7647	1.8885	−1.5432	0.5836
5	1.3050	−1.7595	1.5279	−1.7647	1.6656	−1.7136	1.8885	−1.5432	0.3607
6	1.3050	−1.7595	1.4427	−1.7755	1.5279	−1.7647	1.6656	−1.7136	0.2229
7	1.3050	−1.7595	1.3901	−1.7742	1.4427	−1.7755	1.5279	−1.7647	0.1378
8	1.3901	−1.7742	1.4427	−1.7755	1.4752	−1.7732	1.5279	−1.7647	0.0851

Note that the current minimum is highlighted for every iteration. After the eighth iteration, the minimum occurs at $x = 1.4427$ with a function value of -1.7755. Thus, the result is converging on the true value of -1.7757 at $x = 1.4276$.