8. Review of Linear Algebra

Vector

- 1D array

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{\text{column vector}} \qquad x_i : \text{element, entry, component}$$

$$= [x_1, x_2, \ldots x_n]^T$$

Equality

$$X = Y \quad \text{iff} \quad x_i = y_i \ , 1 \le i \le n$$

Addition/Subtraction

$$X \pm Y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Scalar Product

$$\alpha \cdot X = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} \qquad \alpha : \text{constant or scalar}$$

Linear Combination

Vectors $X^{(1)}, X^{(2)}, \ldots, X^{(m)}$

Scalars $\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_m$

$$\sum_{i=1}^{m} \alpha_i X^{(i)} = \alpha_1 \cdot X^{(1)} + \alpha_2 X^{(2)} + \cdots + \alpha_m X^{(m)} = \begin{bmatrix} \sum_{i=1}^{m} \alpha_i X_1^{(i)} \\ \sum_{i=1}^{m} \alpha_i X_2^{(i)} \\ \vdots \\ \sum_{i=1}^{m} \alpha_i X_n^{(i)} \end{bmatrix}$$

Unit Vectors

$$e^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad e^{(n)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\sum_{i=1}^{n} \alpha_i e^{(i)} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$X = x_1 \cdot e^{(1)} + x_2 e^{(2)} + \cdots + x_n e^{(n)} = \sum_{i=1}^{n} x_i e^{(i)}$$

Vector X is a linear combination of unit vectors.

Ex.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Dot Product of X and Y

$$X^T Y = [x_1\ x_2\ \dots\ x_n] \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i$$

Matrix

- 2D array

$$\underset{m \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix}, \qquad \text{or} \quad A = (a_{ij})$$

Note.

$$\begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} & \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} & \begin{bmatrix} 9 \\ 10 \\ 11 \\ 12 \end{bmatrix} & \begin{bmatrix} 13 \\ 14 \\ 15 \\ 16 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} [1, & 5, & 9, & 13] \\ [2, & 6, & 10, & 14] \\ [3, & 7, & 11, & 15] \\ [4, & 8, & 12, & 16] \end{bmatrix}$$

Square Matrix
$(m = n)$

Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = [\, e^{(1)}, e^{(2)}, \dots e^{(n)} \,]$$

Diagonal Matrix

$$D = \begin{bmatrix} d_{11} & & \bigcirc \\ & d_{12} & \\ \bigcirc & & \ddots & \\ & & & d_{nn} \end{bmatrix} = \text{diag}(d_1, d_2, .. d_n)$$

Tridiagonal Matrix

$$T = \begin{bmatrix} d_1 & c_1 & & & & \bigcirc \\ a_1 & d_2 & c_2 & & & \\ & a_2 & d_3 & c_3 & & \\ & & & \ddots & \ddots & \\ & \bigcirc & & & d_{n-1} & c_{n-1} \\ & & & & a_{n-1} & d_n \end{bmatrix}$$

super diagonal
main diagonal
subdiagonal

Lower Triangular Matrix

$$\begin{pmatrix} 6 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 3 & 5 & 1 & 0 \\ 4 & 8 & 4 & 2 \end{pmatrix}$$

Upper Triangular Matrix

$$\begin{pmatrix} 6 & -1 & 7 & 4 \\ 0 & 7 & 5 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Equality

$$A = B \text{ iff } a_{ij} = b_{ij}, \quad 1 \le i \le n \text{ and } 1 \le j \le n$$

Addition/Subtraction

$$A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1m} \pm b_{1m} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2m} \pm b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} \pm b_{n1} & a_{n2} \pm b_{n2} & \cdots & a_{nm} \pm b_{nm} \end{bmatrix}$$

Scalar Product

$$\alpha \cdot A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \cdots \alpha \cdot a_{1m} \\ \alpha a_{21} & \alpha a_{22} & \cdots \alpha a_{2m} \\ \vdots & \vdots & \\ \alpha a_{n1} & \alpha a_{n2} & \cdots \alpha a_{nm} \end{bmatrix}$$

Matrix Multiplication

$$\mathbf{A}_{m\times n} \quad \mathbf{B}_{n\times p} \quad \mathbf{C}_{p\times q} \stackrel{?}{=} \quad \mathbf{D}$$

(1) Associative

$$(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\,(\mathbf{B}\cdot\mathbf{C})$$

(2) Distributive

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

(3) Commutative?

$$\mathrm{A}\cdot\mathrm{B} = \mathrm{B}\cdot\mathrm{A}\ (?)$$

Transpose

$$\begin{bmatrix} 1 & 4 & 9 \\ 5 & 2 & 3 \\ 10 & 6 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 10 \\ 4 & 2 & 6 \\ 9 & 3 & 1 \end{bmatrix}$$

$$a_{ij} \leftrightarrow a_{ji}$$

Symmetric

$a_{ij} = a_{ji}$ i.e., $A = A^T$

Inverse Matrix

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

Note. A is square and nonsingular.

Ex. A_{2x2}

$$A^{-1} = \frac{1}{a_{11}a_{12} - a_{12} \cdot a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Augmentation

Augmented matrix on A_{3x3}

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Properties

1. $AB \neq BA$ (in general)
2. $A \cdot I = I \cdot A = A$
3. $A \cdot 0 = 0 \cdot A = 0$
4. $(A^T)^T = A$
5. $A \cdot A^{-1} = A^{-1} \cdot A = I$
6. $(A + B)^T = A^T + B^T$
7. $(A \cdot B)^T =$