9 Linear Algebra Problems

1. **Solving system of linear equations** 75%
2. ***Solving eigenvalue problem***

Numerical Linear Algebra

- new discipline

Best reference: Wilkinson and Reinsch (1971)
"Handbook for automatic computation, Vol. 2: Linear Algebra"

$$A_{n\times n}\, x = \underline{b} \qquad \text{n equations, n unknown}$$

$$\begin{cases} a_{11}\cdot x_1 + a_{12}\cdot x_2 + \cdots + a_{1n} x_n = b_1 \\ a_{21}\, x_1 + a_{22}\cdot x_2 + \cdots + a_{2n} x_n = b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{n1}\cdot x_n + a_{n2}\cdot x_n + \cdots + a_{nn} x_n = b_n \end{cases}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Note.

|equations| = |unknowns|		linear system
|equations| > |unknowns|		least square
|equations| < |unknowns|	+ constraints	linear programming

Three Bungee Jumpers Problem

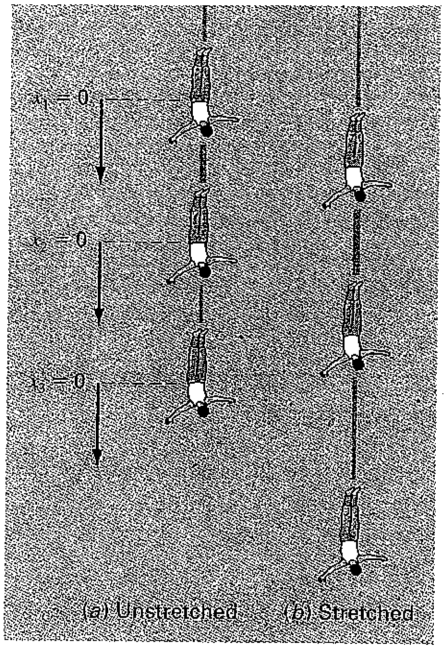


FIGURE 8.1
Three individuals connected by bungee cords.

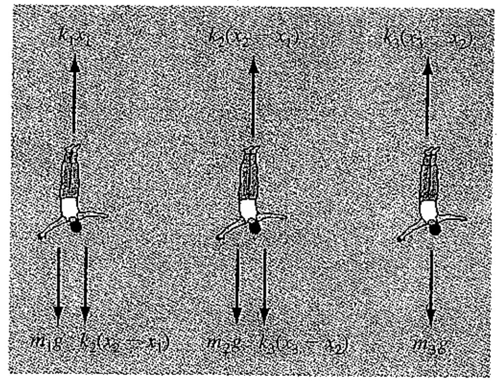


FIGURE 8.2
Free-body diagrams.

$$
\begin{cases}
m_1 g + k_2(x_2 - x_1) - k_1 x_1 = 0 \\
m_2 g + k_3(x_3 - x_2) - k_2(x_2 - x_1) = 0 \\
m_3 g - k_3(x_3 - x_2) = 0
\end{cases}
$$

$$
\begin{cases}
(k_1 + k_2)x_1 - k_2 x_2 = m_1 g \\
-k_2 x_1 + (k_2 + k_3)x_2 - k_3 x_3 = m_2 g \\
-k_3 x_2 + k_3 x_3 = m_3 g
\end{cases}
$$

$$
\begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} m_1 \cdot g \\ m_2 \cdot g \\ m_3 \cdot g \end{bmatrix}
$$

system of linear equations

Three Bungee Jumpers Problem with MATLAB

Jumper	mass	Spring constant	Code length(original)
Top	60	50	20
Middle	70	100	20
Bottom	80	50	20

Matlab solution.

```
>>  K = [150  -100   0;  -100  150  -50;  0 -50  50]
                                        K =   150  -100    0
                                             -100   150  -50
                                                0   -50   50
>>  mg = [588.6;  686.7;  784.8]
                                        mg =  588.6000
                                              686.7000
                                              784.800
>>  X  =  K \ mg   or   X = inv(K) * mg
                                        X =  41.2020
                                             55.9170
                                             71.6130
```

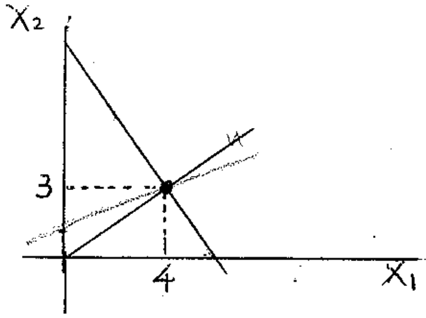
[1] Graphical method

Ex. $3x_1 + 2x_2 = 18$ $\qquad$ $\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 18 \\ 2 \end{bmatrix}$

$-x_1 + 2x_2 = 2$

$x_2 = -3/2\,x + 9$

$x_2 = 1/2\,x + 1$



$\Rightarrow$ $x_1 = 4$, $x_2 = 3$

[2] Cramer's rule

Ex $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

D is the determinant of the matrix A.

[3] Elimination of unknowns

Ex. $$3x_1 + 2x_2 = 18 \quad \text{— (1)}$$
$$-x_1 - 2x_2 = 2 \quad \text{— (2)}$$

(1) x (-1) and (2) x (3)

$$-3x_1 - 2x_2 = -18$$
$$-3x_1 - 6x_2 = 6 \quad (-$$

$$4x_2 = -24$$

$$\boxed{x_2 = -6}$$

From (1), $3x_1 + 2(-6) = 18$

$$3x_1 = 30$$
$$\boxed{x_1 = 10}$$

GAUSS ELIMINATION

Example.

Phase I

$$
\begin{array}{c}
\\ \\ -2 \\ -\frac{1}{2}
\end{array}
\begin{array}{cccc}
x_1 & x_2 & x_3 & b \\
\end{array}
$$

$$
\begin{array}{c}
\\ -2 \\ -\frac{1}{2}
\end{array}
\left(\begin{array}{ccc|c}
6 & -2 & 2 & 16 \\
12 & -8 & 6 & 26 \\
3 & -13 & 9 & -18
\end{array}\right)
$$

$$
\begin{array}{c}
\\ \\ -3
\end{array}
\left(\begin{array}{ccc|c}
6 & -2 & 2 & 16 \\
0 & -4 & 2 & -6 \\
0 & -12 & 8 & -26
\end{array}\right)
$$

$$
\left(\begin{array}{ccc|c}
6 & -2 & 2 & 16 \\
0 & -4 & 2 & -6 \\
0 & 0 & 2 & -8
\end{array}\right)
$$

upper triangular form

Phase II

$$
\begin{array}{rcl}
6x_1 - 2x_2 + 2x_3 &=& 16 \\
-4x_2 + 2x_3 &=& -6 \\
2x_3 &=& -8
\end{array}
\qquad \uparrow \qquad
\begin{array}{l}
x_1 = 23/6 \\
x_2 = -1/2 \\
x_3 = -4
\end{array}
$$

back substitution

Check:

$6x_1 - 2x_2 + 2x_3 = 16$:	$23 + 1 - 8 = 16$	o.k
$12x_1 - 8x_2 + 6x_3 = 26$:	$46 + 4 - 24 = 26$	o.k.
$3x_1 - 13x_2 + 9x_3 = -18$:	$23/2 + 13/2 - 36 = -18$	o.k.

Triangular System

(1) Upper-triangle Case

$$
\begin{array}{rcl}
U_{11} x_1 + \quad \cdots \quad + U_{1,n-1} x_{n-1} + U_{1,n} x_n & = & b_1 \\
\vdots & & \vdots \\
\ddots \qquad \vdots & & \vdots \\
U_{n-1,n-1} x_{n-1} + U_{n-1,n} x_n & = & b_{n-1} \\
U_{n,n} x_n & = & b_n
\end{array}
$$

$$
\begin{cases}
x_n = \dfrac{b_n}{u_{nn}} \\[2ex]
x_{n-1} = \dfrac{b_{n-1} - u_{n-1,n} x_n}{u_{n-1,n-1}} \\[2ex]
\quad \vdots \qquad\qquad \vdots \\
x_1 = \dfrac{b_1 - u_{1,n} x_n - u_{1,n-1} x_{n-1} - \cdots - u_{1,2} x_2}{u_{1,1}}
\end{cases}
$$

$$
\boxed{x_i = \frac{b_i - \sum_{k=i+1}^{n} u_{i,k} \cdot x_k}{u_{ii}}, \quad i = n, n-1, \cdots 1}
$$

(2) Lower-triangle case

$$L x = b$$

$$
\boxed{x_i = \frac{b_i - \sum_{k=1}^{i-1} l_{i,k} x_k}{l_{ii}}, \quad i = 1, 2, \cdots n}
$$

- Frequency count
  - ① division: n
  - ② add/mul: $\sum_{i=1}^{n} (i-1) \cdot 2 = n \cdot (n-1)$

    $= n^2 + n \quad : \quad O(n^2)$

Frequency Count

(Phase I)

step k: (n – k) divisions, (n - k)(n – k + 1) multiplications/additions

If we ignore divisions,

$$\sum_{k=1}^{n-1} 2\,(n-k)\,(n-k+1)$$

$$= 2/3\; n\,(n^2 - 1) + O(n^2)$$

$$= \underline{2/3\; n^3} + O(n^2)$$

(Phase II)

$\underline{n^2}$

Total number of operations: $2/3\; n^3 + n^2$

TABLE 9.1 Number of flops for naive Gauss elimination.

n	Elimination	Back Substitution	Total Flops	$2n^3/3$	Percent Due to Elimination
10	705	100	805	667	87.58%
100	671550	10000	681550	666667	98.53%
1000	6.67×10^8	1×10^6	6.68×10^8	6.67×10^8	99.85%

Pivoting Strategy

Motivation.

Ex.1 (When the pivot element is zero)

$$\begin{cases} x_1 + x_2 + x_3 = 1 & (1) \\ x_1 + x_2 + 2x_3 = 2 & (2) \\ x_1 + 2x_2 + 2x_3 = 1 & (3) \end{cases}$$

$$(x_1 = 1, x_2 = -1, x_3 = 1)$$

After elimination step,

$$\begin{cases} \boxed{0}\, x_2 + x_3 = 1 & (2)' \\ x_2 + x_3 = 0 & (3)' \end{cases}$$

Note. The pivot element $a_{22}^{(2)}$ is zero.

If we interchange equations (2)' and (3)',

$$\begin{cases} x_2 + x_3 = 0 \\ x_3 = 1 \end{cases} \qquad \begin{array}{l} x_2 = -1 \\ x_3 = 1 \end{array}$$

Ex. 2 (When the pivot element is nearly zero)

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + 1.0001\,x_2 + 2x_3 = 2 \\ x_1 + 2\,x_2 + 2x_3 = 1 \end{cases}$$

[Case 1] Without Pivoting

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 0.0001\,x_2 + x_3 = 1 \\ x_2 + x_3 = 0 \end{cases}$$

$$\begin{cases} x + x_2 + x_3 = 1 \\ 0.0001\,x_2 + x_3 = 1 \\ -9999\,x_3 = -10000 \end{cases} \qquad \begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 1 \end{aligned}$$

: wrong answer

[Case 2] With Pivoting

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 0.0001\,x_2 + x_3 = 1 \\ x_2 + x_3 = 0 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_2 + x_3 = 0 \\ 0.0001\,x_2 + x_3 = 1 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_2 + x_3 = 0 \\ 0.9999\,x_3 = 1 \end{cases} \qquad \begin{aligned} x &= 1 \\ x &= -1 \\ x &= 1 \end{aligned}$$

: correct answer

```
function x = GaussPivot(A,b)
% GaussPivot: Gauss elimination pivoting
%   x = GaussPivot(A,b): Gauss elimination with pivoting.
% input:
%   A = coefficient matrix
%   b = right hand side vector
% output:
%   x = solution vector

[m,n]=size(A);
if m~=n, error('Matrix A must be square'); end
nb=n+1;
Aug=[A b];
% forward elimination
for k = 1:n-1
  % partial pivoting
  [big,i]=max(abs(Aug(k:n,k)));
  ipr=i+k-1;
  if ipr~=k
    Aug([k,ipr],:)=Aug([ipr,k],:);
  end
  for i = k+1:n
    factor=Aug(i,k)/Aug(k,k);
    Aug(i,k:nb)=Aug(i,k:nb)-factor*Aug(k,k:nb);
  end
end
% back substitution
x=zeros(n,1);
x(n)=Aug(n,nb)/Aug(n,n);
for i = n-1:-1:1
  x(i)=(Aug(i,nb)-Aug(i,i+1:n)*x(i+1:n))/Aug(i,i);
end
```

FIGURE 9.5
An M-file to implement the Gauss elimination with partial pivoting.

Cases where pivoting is not needed.

(1) Diagonally dominent.

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad 1\leq i\leq n$$

Tridiagonal case: $|d_i| > |c_i| + |a_{i-1}| \quad 1\leq i\leq n$

(2) Symmetric and positive-definite.

$$A^T = A \quad \text{and} \quad x^T A x > 0 \quad \text{for all } x \neq 0.$$

Tridiagonal System

– special case of band structure

$$\begin{bmatrix} d_1 & c_1 & & & 0 \\ a_1 & d_2 & c_2 & & \\ & a_2 & d_3 & c_3 & \\ & & \ddots & \ddots & c_{n-1} \\ 0 & & & a_{n-1} & d_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

$$a_{ij} = 0 \quad \text{if } |i-j| \geq 2$$

$$\text{Space}(n) = n + (n-1) + (n-2) = 3n-2$$

Forward Elimination phase

$$\begin{cases} d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}}\right) \cdot c_{i-1} \\ b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}}\right) \cdot b_{i-1} \qquad 2 \leq i \leq n \end{cases}$$

Backward Substitution phase

$$\begin{cases} x_n \leftarrow b_n / d_n \\ x_i \leftarrow \dfrac{b_i - c_i \cdot x_{i+1}}{d_i} \qquad n-1 \geq i \geq 1 \end{cases}$$

$$\begin{bmatrix} 2.04 & -1 & & \\ -1 & 2.04 & -1 & \\ & -1 & 2.04 & -1 \\ & & -1 & 2.04 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{Bmatrix}$$

$$f_2 = f_2 - \frac{e_2}{f_1} g_1 = 2.04 - \frac{-1}{2.04}(-1) = 1.550$$

$$r_2 = r_2 - \frac{e_2}{f_1} r_1 = 0.8 - \frac{-1}{2.04}(40.8) = 20.8$$

$$\begin{bmatrix} 2.04 & -1 & & \\ & 1.550 & -1 & \\ & & 1.395 & -1 \\ & & & 1.323 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 40.8 \\ 20.8 \\ 14.221 \\ 210.996 \end{Bmatrix}$$

$$x_4 = \frac{r_4}{f_4} = \frac{210.996}{1.323} = 159.480$$

$$x_3 = \frac{r_3 - g_3 x_4}{f_3} = \frac{14.221 - (-1)159.480}{1.395} = 124.538$$

$$x_2 = \frac{r_2 - g_2 x_3}{f_2} = \frac{20.800 - (-1)124.538}{1.550} = 93.778$$

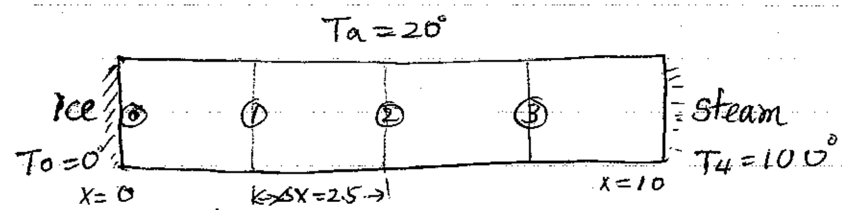
$$x_1 = \frac{r_1 - g_1 x_2}{f_1} = \frac{40.800 - (-1)93.778}{2.040} = 65.970$$

```
function x = Tridiag(e,f,g,r)
% Tridiag: Tridiagonal equation solver banded system
%   x = Tridiag(e,f,g,r): Tridiagonal system solver.
% input:
%   e = subdiagonal vector
%   f = diagonal vector
%   g = superdiagonal vector
%   r = right hand side vector
% output:
%   x = solution vector
n=length(f);
% forward elimination
for k = 2:n
  factor = e(k)/f(k-1);
  f(k) = f(k) - factor*g(k-1);
  r(k) = r(k) - factor*r(k-1);
end
% back substitution
x(n) = r(n)/f(n);
for k = n-1:-1:1
  x(k) = (r(k)-g(k)*x(k+1))/f(k);
end
```

FIGURE 9.6
An M-file to solve a tridiagonal system.

Steady-state Heated Rod

problem



What are the temperatures at ①, ②, and ③?

Differential equation:

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0$$

where

$T(x)$: temperature at x

h' : heat transfer coefficient

T_a : air temperature

Numerical Solution

Differential equation $\Rightarrow$ Difference equation:

From Section 4.3,

$$f''(x) \cong \frac{\frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f(x_i) - f(x_{i-1})}{h}}{h}$$

$$\frac{d^2T}{dx} = \frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2}$$

$$\therefore \boxed{\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} + h'(T_a - T_i) = 0}$$

$\Delta x = 2.5$

$h' = 0.01$

$T(a) = 20$

Change T_{i+1} and T_{i-1}

$$\frac{T_{i+1} - 2T_i + T_{i+1}}{6.25} + 0.01\left(20 - T_i\right) = 0$$

$$T_{i+1} - 2T_i + T_{i+1} + 0.0625\,(20 - T_i) = 0$$

$$T_{i-1} - 2T_i + T_{i+1} + 1.25 - 0.0625\,T_i = 0$$

$$\boxed{T_{i-1} - 2.0625T_i + T_{i+1} = -1.25}$$

$$
\begin{array}{lllll}
i=1 & \overset{=0}{T_0} - 2.0625T_1 + T_2 & & & = -1.25 \\
i=2 & & T_1 - 2.0625T_2 + T_3 & & = -1.25 \\
i=3 & & & T_2 - 2.0625T_3 + \underset{=100}{T_4} & = -1.25
\end{array}
$$

$$
\begin{array}{llll}
-2.0625T_1 + T_2 & & & = -1.25 \\
& T_1 - 2.0625T_2 + T_3 & & = -1.25 \\
& & T_2 - 2.0625T_3 & = -101.25
\end{array}
$$

$$
\begin{bmatrix} -2.0625 & 1 & 0 \\ 1 & -2.0625 & 1 \\ 0 & 1 & -2.0625 \end{bmatrix}
\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}
=
\begin{bmatrix} -1.25 \\ -1.25 \\ -101.25 \end{bmatrix}
$$

Tridiagonal system

```
>> A = [-2.0625   1        0
         1       -2.0625   1
         0        1       -2.0625]

>> b = [-1.25 -1.25 -101.25]'

>> T = inv(A) * b
```

23.2099, 46.6205 71.6948