Vectors and Computer Graphics
CS 425
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1. Introduction

There are many uses for vectors in computer graphics, so it is very important that you have an understanding of what they do for us, as compared to just knowing the definitions. For example, two very common applications are:

- Given two vectors use dot product (which will be in the graphics processor) to find the cosine of the angle between them. Cosine is used extensively in computer graphics, for example in lighting calculations and in back face culling, which we'll see as the semester progresses.

- Given three points in a plane, where it is known whether they were entered clockwise or counterclockwise, compute the unit surface normal in the correct orientation and from this compute the planar equation.

Throughout these notes I'll assume that we are using a right handed system (RHS). For a long time some graphics systems used left handed systems and others used right handed, but now the right handed vector system is used in nearly all graphics software and literature, so it is the obvious assumption to make. To understand the RHS hold up your right hand with thumb and the first two fingers at right angles to each other, then with thumb as X and forefinger as Y, the Z direction is given by the middle finger. A left handed system would have Z in the other direction – try both without breaking any fingers.

Using the older (but equivalent) definition, align your thumb along the X axis, as before, and curl your four fingers together. They will curl in the Z direction under either RHS or LHS.

2. Definitions

I'll give the basic terminology definitions in this section and will leave the definitions of dot product and cross product to the next two sections.

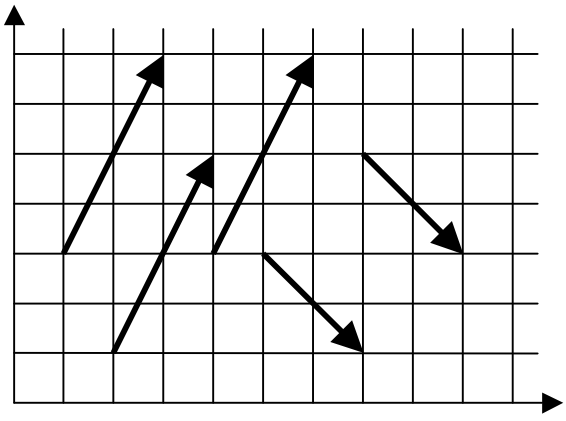
A *vector* is the difference between two points in space. This can be in 2D or 3D (or higher dimensions), but obviously in graphics we are mainly interested in 3D, so I'll usually assume 3D.

E.g., $P_1 = (3, 5, -2)$, $P_2 = (-1, 0, 4)$. The vector **v** with head at P_1 and tail at P_2 is defined by

$$\mathbf{v} = P_1 - P_2 = (4, 5, -6)$$

The elements v_1, v_2, and v_3 which make up the vector $\mathbf{v} = (v_1, v_2, v_3)$ are called its *scalar components*.

It is important to recognize that the definition gives a vector a size and a direction, but that it doesn't give it a location, and it is critical that you understand this concept. So, for example, decide how many 2D vectors I've shown as bold arrows in the figure below.



The temptation is to say that there are five, but actually there are only two. The three on the left from (1, 3) to (3, 7), from (2, 1) to (4, 5), and from (4, 3) to (6, 7) are all the same vector, (2, 4). Similarly the other two vectors are both the same vector, (2, -2). I.e., it isn't that the three arrows on the left represent equal vectors, but they represent the same vector. So, to emphasize this again, a vector has size and direction, but not location.[1]

[1] As we'll see, there is a special vector, the zero vector or just **0**, which has a size but no well defined direction, but we make every effort to avoid it in our graphics computations.

There are a number of ways to distinguish a vector from a *scalar* (a number or variable). In print a vector is usually bold (e.g. **v**) and when handwritten it is either underlined (e.g. $\underline{v}$) or with some kind of arrow or half arrow above it. I'll use bold for vectors in these notes and underline when I'm writing in class. Graphically a vector will be shown as an arrow in some coordinate system.

$\mathbf{v} = (v_1, v_2, v_3)$ is a 3D vector with three scalar components.

The *direction of a vector* can be found by placing the tail on the origin, and so the head will then be at the Cartesian point (v_1, v_2, v_3). (I.e., $P_1 - P_2$ where P_2 is (0, 0, 0).)

The *length/size/magnitude of a vector*, $\|\mathbf{v}\|$, is $\sqrt{v_1^2 + v_2^2 + v_3^2}$.

If $\|\mathbf{v}\| = 1$, then **v** is a **unit vector**.

The *zero vector*, $\mathbf{0} = (0, 0, 0)$, must be avoided at all costs. Note that it has a magnitude (zero), but no well defined direction. If it gets created and you don't detect it then horrible problems can occur. E.g., if you try to find the normal vector to a plane defined by three points, and they happen to be collinear, then the zero vector will be returned since the plane isn't well defined.

Vector addition: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

Scalar multiplication: If a is a scalar and **v** is a vector, then the scalar multiplication of **v** by a is computed as $a\mathbf{v} = (av_1, av_2, av_3)$. If $a > 0$ then you get a vector in the same direction but a times as large, if $a < 0$ then you get a vector that is in the opposite direction and is $|a|$ (i.e. $-a$) times as large. If $a = 0$ you've just created **0** and you are likely to have problems.

Normalizing a vector: If we want a unit vector in the same direction as **v** (which we often do) then we just need to divide the vector by its length (i.e., scalar multiplication by the inverse of the length). So to normalize **v** just compute $\mathbf{v} / \|\mathbf{v}\|$. (Note that unpleasant things will happen if **v** is **0**.)

3. Vector dot product

The dot product is also called *vector scalar product*, but that tends to get confused with scalar multiplication, so I'll use dot product as the name.

$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos(\theta)$, where θ is the angle between the two vectors (stick their tails together).

Computationally, $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

E.g., the dot product of the vectors (3, -1, 2) and (1, 2, -1) is -1 because $3 * 1 + (-1) * 2 + 2 * (-1) = 3 - 2 - 2 = -1$.

As a special case, if **u** and **v** are unit vectors then the angle between them is given by $\cos^{-1}(\mathbf{u} \cdot \mathbf{v})$.

Usually vectors are normalized in graphics, so the dot product lets us compute the cosine of the angle between two vectors very quickly.

4. Vector cross product (right handed)

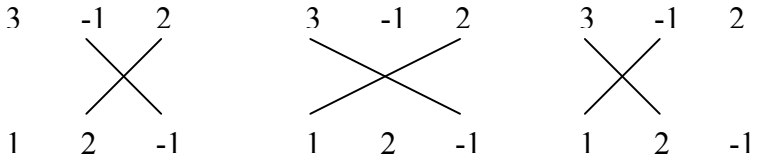
The cross product is also just called the *vector product*, but I'll call it the cross product.

u x **v** = ||**u**|| ||**v**|| sin(θ) **n**, where **n** is a unit vector that is normal to the plane that contains the vectors **u** and **v**, and θ is the angle between **u** and **v**.

Computationally $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$. As I mentioned, above, I'll always use the right handed cross product, but if you want the left handed cross product just change the sign of this expression.

E.g., the cross product of (3, -1, 2) and (1, 2, -1) is (-3, 5, 7) because the formula gives ((-1) * (-1) – 2 * 2, 2 * 1 – 3 * (-1), 3 * 2 – (-1) * 1) which is (1 – 4, 2 + 3, 6 + 1). Practice some cross products because you'll be doing them in homeworks and tests.

I don't bother to remember the formula but remember the crosses that give the operation its name. I'll show what I mean below using the example above:

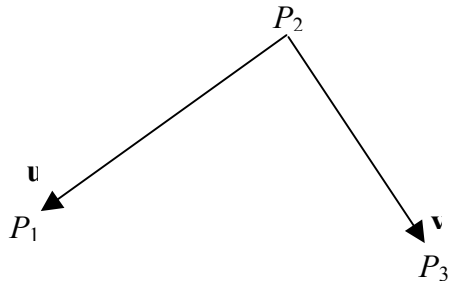


Start in the middle at the top and do the X shown missing the first term, then start at the top right and do an X missing the second term, and finally start at the top left and do an X missing the last term. Compare this against the computation ((-1) * (-1) – 2 * 2, 2 * 1 – 3 * (-1), 3 * 2 – (-1) * 1) which I had above.

So the result of a cross product between two vectors is another vector (with magnitude ||**u**|| ||**v**|| sin(θ)) that is normal (perpendicular) to the plane that contains the two vectors.

The most common use of cross product in graphics is to find the unit normal vector to a plane defined by three points, P_1, P_2, and P_3. Say you know that these three points were entered in that order clockwise in the plane. Define

$\mathbf{u} = P_1 - P_2$ and $\mathbf{v} = P_3 - P_2$. Then (see the picture below) take the cross product $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, and $\mathbf{w}$ will be a normal vector to the plane in the correct direction. Now normalize $\mathbf{w}$ (i.e., assign $\mathbf{w} = \mathbf{w} / ||\mathbf{w}||$) and you've got a unit normal vector to the plane in the correct direction.



To check the direction, lay your thumb (right hand) along $\mathbf{u}$ and your forefinger along $\mathbf{v}$, and then your middle finger should point out of the page, which is what we want.

I'll do an example of this later in these notes.

Note that if $\mathbf{u} = k\mathbf{v}$ for some k, then $\mathbf{u} \times \mathbf{v} = (0, 0, 0) = \mathbf{0}$, the zero vector. This can be seen either from the computational form of the definition or from noting that $\mathbf{u}$ and $\mathbf{v}$ are parallel, that the angle between parallel lines is 0°, and that $\sin(0°) = 0$.

5. Planar equations and normal vectors.

If we know the planar equation is, say, $Ax + By + Cz + D = 0$, then an incredibly useful property is that a normal vector to the plane is (A, B, C). E.g. the plane $3x - y + z + 2 = 0$ has a normal vector (3, -1, 1). (All that the D value is doing is moving the plane up and down perpendicular to this vector.) If we want the two unit normal vectors to the plane, in opposite directions, they are $(\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}})$ and $(\frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}})$.

In the other direction, if we've computed the normal vector as discussed earlier, and also know a point in the plane, we can use this property to compute the planar equation. E.g., a plane contains the point (1, -1, 2) and has a normal vector (2, 1, -3). Then the planar equation must be

$$2x + y - 3z + D = 0$$

for some D. Using the known value on the plane (1, -1, 2) for (x, y, z) gives $2 - 1 - 6 + D = 0$, and so $D = 5$ and the equation of the plane is

$$2x + y - 3z + 5 = 0.$$

6. Example 1: Computing a cosine

Say that we want the cosine of the angle between the two vectors (3, -1, 2) and (1, 2, -1), when we put their tails together. Call the angle θ, the first vector **u** and the second vector **v**. Then we have:

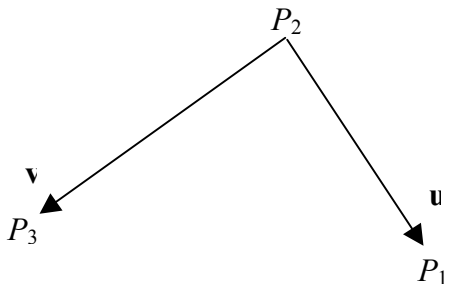
$$\mathbf{u} \cdot \mathbf{v} = 3 - 2 - 2 = -1 = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos(\theta) = \sqrt{14} * \sqrt{6} * \cos(\theta).$$

So $\theta = \cos^{-1}(\frac{-1}{\sqrt{84}}) = 96.3°$.

7. Example 2: Planar Equation from Three Points

A common situation is that a user will enter a number of points, usually counter clockwise (ccw), which define a polygon. The graphics package will computer the planar equation from the first three points that are not collinear, and will check that subsequent points lie on this plane. The package will also usually compute and retain the up vector for the plane and will use this later for lighting calculations and for backface culling.

Earlier I gave a brief example where three points were entered clockwise. In this example I'll assume that they are entered counterclockwise (ccw), which will give a picture like:



where, as before, $\mathbf{u} = P_1 - P_2$ and $\mathbf{v} = P_3 - P_2$. Now to get the up vector coming out of the page we'll need $\mathbf{v} \times \mathbf{u}$ (use your right hand to confirm this).

Say that the three points entered are, in ccw order, (2, 2, -2), (1, 2, -1), and (1, 3, 0), and that we want the planar equation of the plane that contains these three points and a unit up vector to the plane. First I'll compute the up vector, then use it to get the planar equation, and finally normalize it to get the unit up vector.

$\mathbf{u} = P_1 - P_2 = (2, 2, -2) - (1, 2, -1) = (1, 0, -1)$
$\mathbf{v} = P_3 - P_2 = (1, 3, 0) - (1, 2, -1) = (0, 1, 1)$
$\mathbf{v} \times \mathbf{u} = (0, 1, 1) \times (1, 0, -1) = (-1, 1, -1).$

So (-1, 1, -1) is a (not unit) up vector to this plane.

As I discussed earlier we can use this to say that the planar equation must be $-x + y - z + D = 0$, for some D. To find D substitute in any of the three

points, say (1, 3, 0), for (x, y, z). This gives -1 + 3 – 0 + D = 0, so D = -2, which gives the planar equation:

$$-x + y - z - 2 = 0.$$

To confirm that I haven't made an error along the way, substitute the other two points, (2, 2, -2) and (1, 2, -1), into this equation, and you'll find that it is satisfied for them.

My final task is to find the unit up vector, which we get by normalizing the up vector (-1, 1, -1). The length of this vector is $\sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}$, so the unit up vector is $(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}})$. If we want the unit vector in the down direction we just have to change the signs, giving $(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

8. Example 3: Trying to Define a Plane with Collinear Points

Say that the three points entered by the user in ccw order are (2, 3, 1), (1, 2, -1), and (-1, 0, -5). Defining **u** and **v** as before we get:

$$\mathbf{u} = (2, 3, 1) - (1, 2, -1) = (1, 1, 2)$$
$$\mathbf{v} = (-1, 0, -5) - (1, 2, -1) = (-2, -2, -4)$$
$$\mathbf{v} \times \mathbf{u} = (-2, -2, -4) \times (1, 1, 2) = (0, 0, 0) = \mathbf{0}.$$

So the up vector to the plane is the zero vector, which has no well defined direction, and which will cause chaos with later display calculations. So what has gone wrong? If we look more carefully at **u** and **v** we can see that $\mathbf{v} = -2\mathbf{u}$, and so they are parallel (in opposite directions). This means that our three points are collinear[2], and obviously a line cannot define a unique plane.

This is very useful as long as the graphics package detects the problem. The procedure should be that the user puts in three or more points that they claim define a polygon. The graphics package uses the first three points to get the normal vector, and if it isn't the zero vector it can then get the planar equation and check to ensure that any other points lie in the plane. If, however, it gets the zero vector then it tries using points two through four to repeat the process. It keeps going until finally it gets three non-collinear points or it runs out of points and so determines that all of the entered points were collinear (in which case it can output an error message and exit).

[2] lie in a straight line