Computing the Nearest Neighbor Transform Exactly with only Double Precision

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Timothy M. Chan Jack Snoeyink

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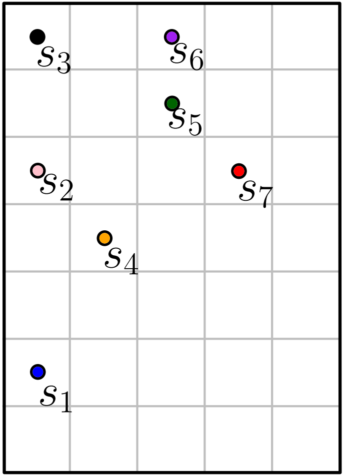
The Problem



Given n sites on a pixel grid,
what is the closest site to each pixel?

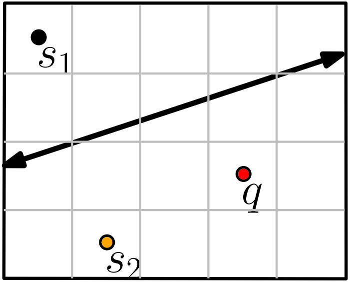
How much precision is need to determine this?

Other precision/robust approaches

Techniques for implementing geometric algorithms with finite precision computer arithmetic:

- Rely on machine precision (+epsilon)
- Topological Consistency [S99, S01, SI90, SI92, SII*00]
- Exact Geometric Computation [Y97]
  - Software based arithmetic [CORE, LEDA, MPFR]
  - Predicate eval. schemes [C92, FW93, ABO*97, S97,GRR00]
  - Degree-driven algorithm design [LPT99]

Is q closer to s_1?



$$
\begin{aligned}
\mathbb{U} &= \{1, 2, \ldots, U\} \\
s_1, \quad s_2, & \quad q \in \mathbb{U}^2 \\
s_1 &= (x_1, y_1) \\
s_2 &= (x_2, y_2) \\
q &= (x_q, y_q)
\end{aligned}
$$

$$\|q - s_1\|^2 \quad \gtrless \quad \|q - s_2\|^2$$

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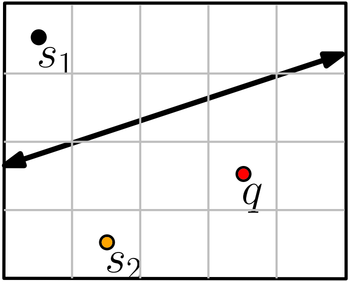

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$$f(s_1, s_2, q) = \operatorname{sign}((x_q - x_1)^2 + (y_q - y_1)^2 - (x_q - x_2)^2 - (y_q - y_2)^2)$$

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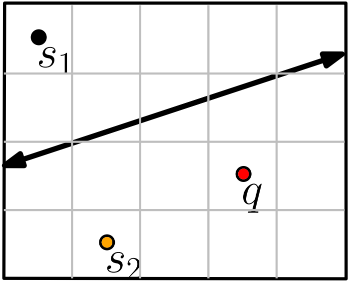

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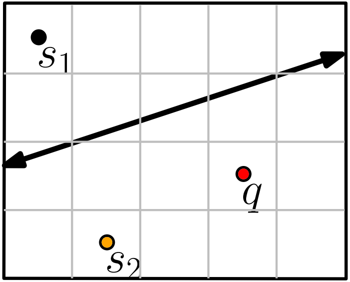

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f(s_1, s_2, q) &= \mathrm{sign}((x_q - x_1)^2 + (y_q - y_1)^2 - (x_q - x_2)^2 - (y_q - y_2)^2) \\
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&= \textcircled{2}
\end{aligned}
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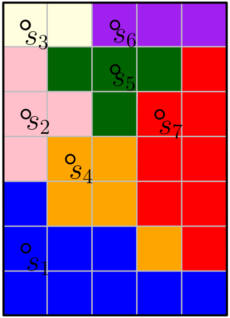


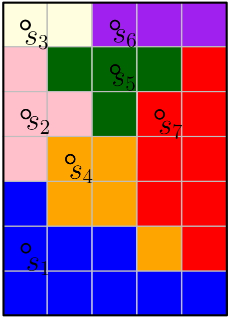
Given

A grid of size U and
Sites $S = \{s_1, \ldots, s_n\} \subset \mathbb{U}^2$

Label

Each grid point of $\mathbb{U}^2$ with the closest site of S

Nearest Neighbor Transform



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	Alg	Time
Brute Force	deg 2	$O(nU^2)$
Nearest Neighbor Trans. [B90]	deg 5	$O(U^2)$
Discrete Voronoi diagram [C06, MQR03]	deg 3	$O(U^2)$
GPU Hardware [H99]	-	$\Theta(nU^2)$

Problem Transformations–Part 1

Problem (NNTrans-min)

For each pixel $q \in U^2$, find the site $s_i \in S$ such that,
for all $j < i$, the distance $\|q - s_i\| < \|q - s_j\|$, and
for all $j > i$, the distance $\|q - s_i\| \leq \|q - s_j\|$.

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$$\|q - s_i\|^2 < \|q - s_j\|^2$$
$$q \cdot q - 2q \cdot s_i + s_i \cdot s_i < q \cdot q - 2q \cdot s_j + s_j \cdot s_j$$
$$2x_i x_q + 2y_i y_q - x_i^2 - y_i^2 > 2x_j x_q + 2y_j y_q - x_j^2 - y_j^2.$$

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Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ that achieves the maximum of $2x_i x_q + 2y_i y_q - x_i^2 - y_i^2$.

Problem Transformations–Part 2

Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ *that achieves the maximum of* $2x_i x_q + 2y_i y_q - x_i^2 - y_i^2$.

Problem Transformations–Part 2

Problem (NNTrans-max)

For each pixel q, find the site with lowest index $s_i \in S$ that achieves the maximum of $2x_ix_q + 2y_iy_q - x_i^2 - y_i^2$.

For a fixed row, $y_q = Y$

$$2x_ix_q + 2y_iy_q - x_i^2 - y_i^2 > 2x_jx_q + 2y_jy_q - x_j^2 - y_j^2$$
$$2x_ix_q + (2y_iY - x_i^2 - y_i^2) > 2x_jx_q + (2y_jY - x_j^2 - y_j^2)$$
$$①x_q + ② > ①x_q + ②$$

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$$①x_q + ② > ①x_q + ②$$

Problem (DUE-Y)

For a fixed $1 \leq Y \leq U$, and for each $1 \leq X \leq U$, find the smallest index of a line of L_Y with maximum y coordinate at $x = X$.

Problem Transformations–Part 2

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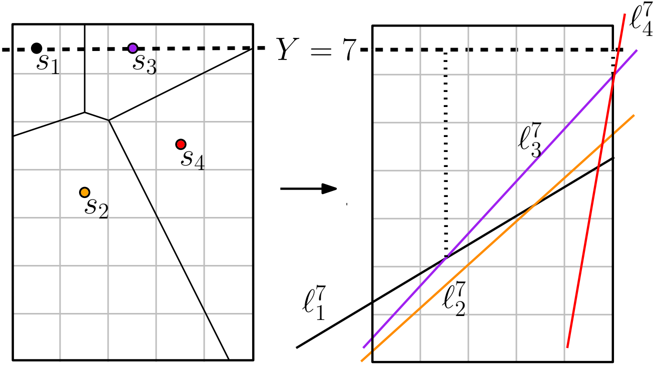
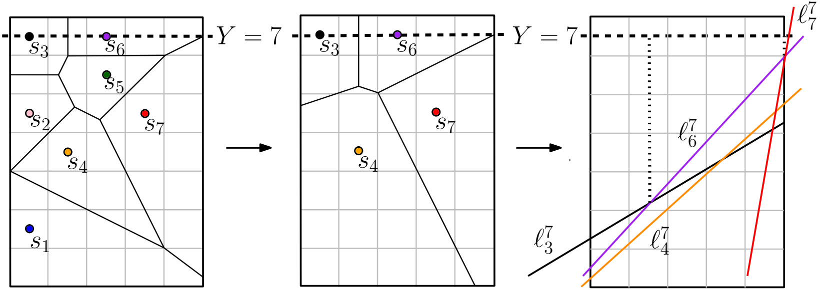


Sketch of NNTransform Algorithm



Three Algorithms for Computing the DUE

Discrete Upper Envelope Lemma

Given a set of n lines L of each of the form $y = ① x + ②$ we can compute the DUE of L in:

- $O(n + U)$ and degree 3, (`DUE-DEG3`)
- $O(n + U \log U)$ and degree 2, (`DUE-ULgU`)
- $O(n + U)$ expected time and degree 2. (`DUE-U`)

For each item:

1. Reduce L to at most U lines.
2. Compute DUE of at most U lines.

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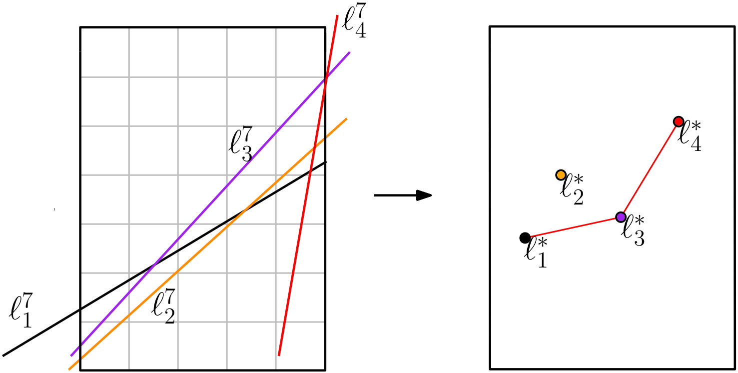
Discrete Upper Envelope Lemma (DUE-DEG3)

Two steps:

1. Compute upper env. via the lower hull of dual points
   line $y = mx + b$ maps to dual point $(m, -b)$.
2. Discretize upper env to DUE.



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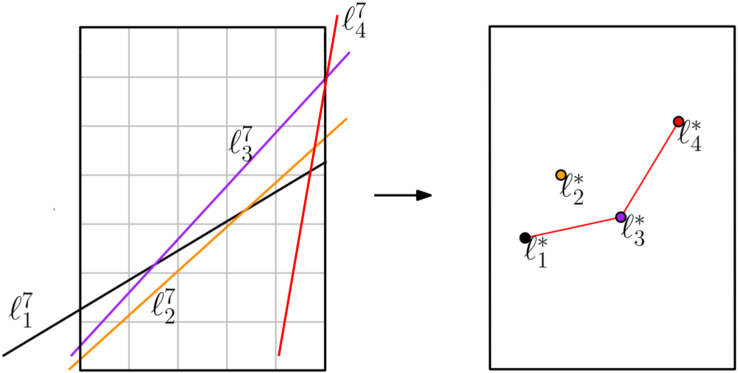
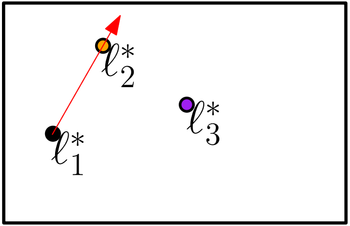


line form is $y = ①x + ② \implies$ dual point form is $(①, ②)$

Orientation test



ℓ_1^*, ℓ_2^* and ℓ_3^* have form (①, ②)

$$\texttt{orient}(\ell_1^*, \ell_2^*, \ell_3^*) = \operatorname{sign}\left(\begin{vmatrix} ⓪ & ① & ② \\ ⓪ & ① & ② \\ ⓪ & ① & ② \end{vmatrix}\right) = \operatorname{sign}(③)$$

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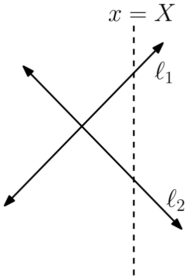
Main predicate `OrderOnALine`

Order on a Line



$X = ①$
ℓ_1, ℓ_2 have form $y = ①x + ②$

$$\texttt{orderOnLine}(\ell_1, \ell_2, X) = \mathrm{sign}(①X + ② - ①X + ②) = \mathrm{sign}(②)$$

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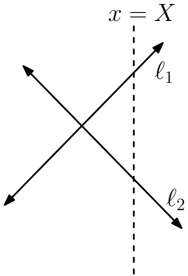

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Lemma `IntersectCol`

Given two lines ℓ_1 and ℓ_2 of the form $y = ①x + ②$, the construction $\texttt{IntersectCol}(\ell_1, \ell_2)$ returns the column containing the intersection of the two lines in $O(logU)$ time and degree 2.

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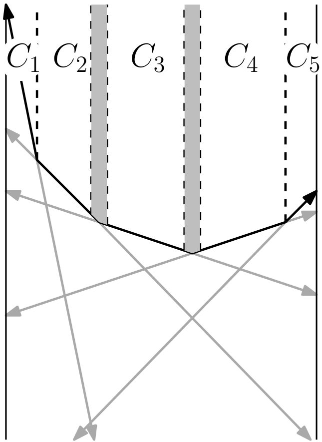
Discrete Upper Envelope Lemma (DUE-ULgU)



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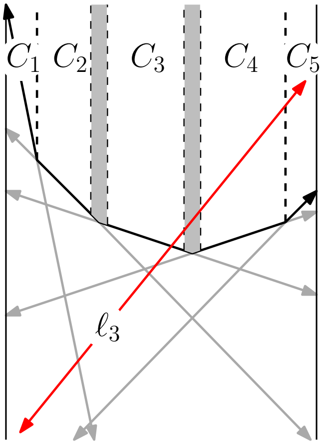

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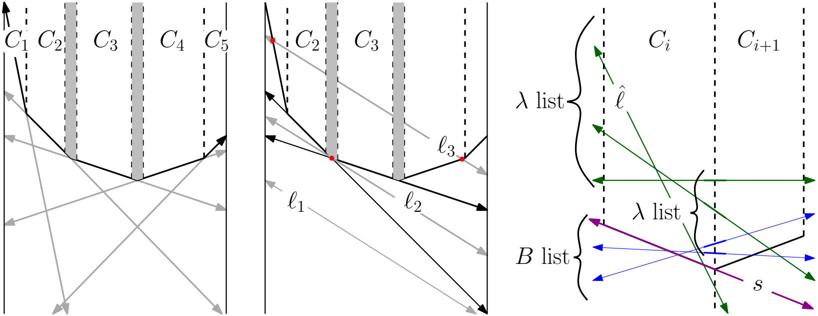
Discrete Upper Envelope Lemma (DUE-U)



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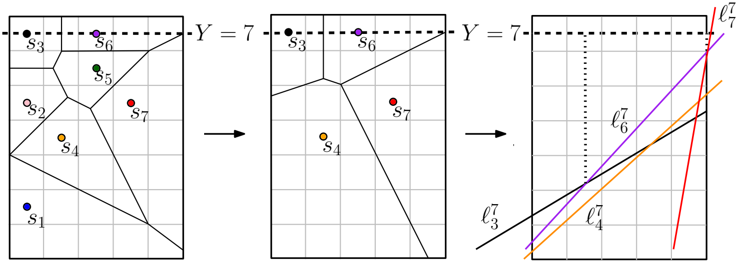
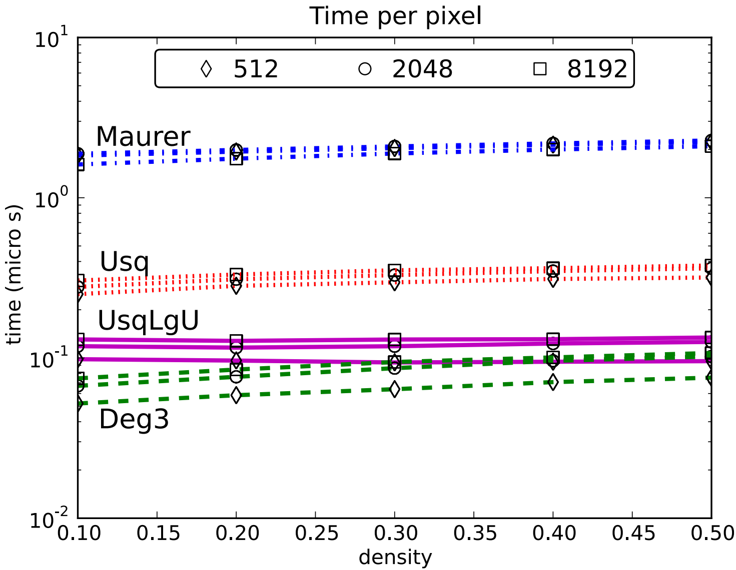
NNTransorm Theorem

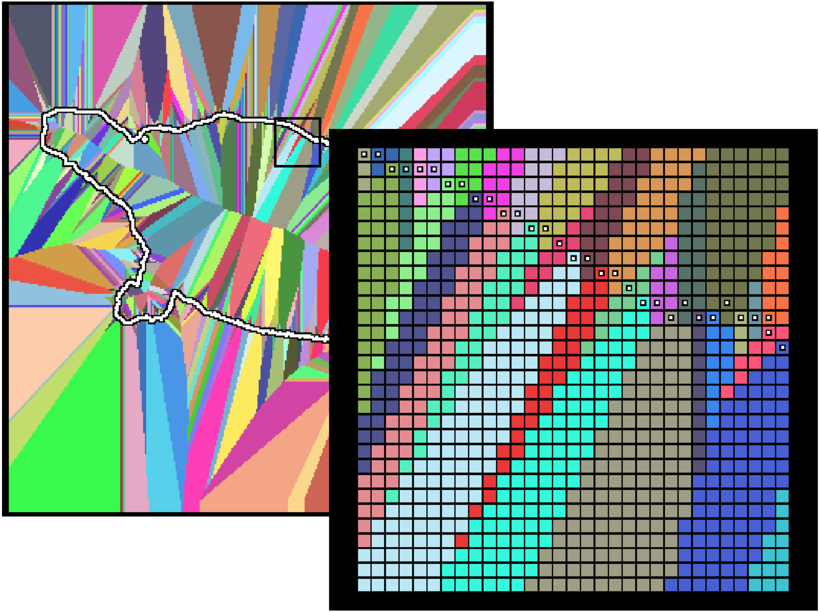
Given a set of n sites with degree 1 coordinates on a $U \times U$ grid, we can compute the nearest neighbor transform in:

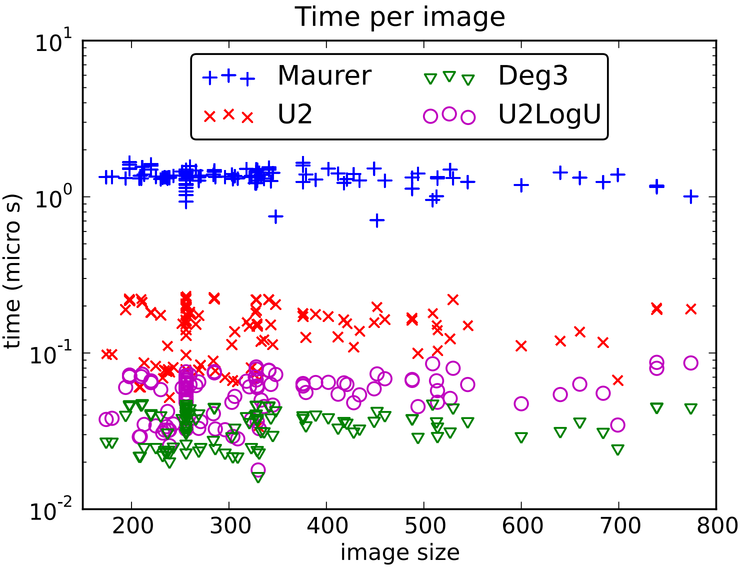
- $O(U^2)$ and degree 3, (Deg3)
- $O(U^2 \log U)$ and degree 2, (UsqLgU)
- $O(U^2)$ expected time and degree 2. (Usq)




Time per pixel
◊ 512
○ 2048
□ 8192
Maurer
Usq
UsqLgU
Deg3
time (micro s)
density
10^1
10^0
10^{-1}
10^{-2}
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50

Experiments Part 2




Time per image
Maurer
Deg3
U2
U2LogU
time (micro s)
10^{1}
10^{0}
10^{-1}
10^{-2}
200
300
400
500
600
700
800
image size

Described and implemented three algorithms
for computing the DUE of lines of the form $y = ①x + ②$:

- `DUE-DEG3`: $O(n + U)$ and degree 3
- `DUE-ULgU`: $O(n + U \log U)$ and degree 2
- `DUE-U`: $O(n + U)$ expected time and degree 2

Which gave us three algorithms for computing the NNTransform:

- `Deg3`: $O(U^2)$ and degree 3
- `UsqLgU`: $O(U^2 \log U)$ and degree 2
- `Usq`: $O(U^2)$ expected time and degree 2.

Conclusions

Can we compute the NNTraform with degree 2 without randomization?

What about L_1 or L_∞?

What other geometric problems can be considered using degree-driven algorithm design?

Summary

Described and implemented three algorithms
for computing the DUE of lines of the form $y = ① x + deg(2)$:

- `DUE-DEG3`: $O(n + U)$ and degree 3
- `DUE-ULgU`: $O(n + U \log U)$ and degree 2
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Which gave us three algorithms for computing the NNTransform:

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- `UsqLgU`: $O(U^2 \log U)$ and degree 2
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Contact

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web: `http://cs.unc.edu/~dave`