To prove a language $L$ is not context free:

Show that for any $p$ there is a string $s \in L$ with $|s| \geq p$ such that for any $uvxyz = s$ with $|vxy| \leq p$ and $|vy| > 0$, there exists an $i$ such that $uv^i xy^i z \notin L$

Step 1: provide a candidate string $s$ within $L$ that is longer than $p$

- Make sure $s \in L$ and $|s| \geq p$ for any $p$

Step 2: show that if $uvxyz = s$, $|vxy| \leq p$, and $|vy| > 0$, then $uvxyz$ must have certain characteristics or falls into certain cases.

- If you use cases, make sure they are exhaustive and broad
- In general, focus on $vxy$ and $|vxy| \leq p$

Step 3: show that there is some $i$ such that $uv^i xy^i z \notin L$

- in many cases you can use $i = 0$ or $i = 2$ will work
- if you used cases in step two, show that for any case you can make $uv^i xy^i z \notin L$
Example 1: \( \{a^{n^2} : n \geq 0 \} \)

Step 1: finding a candidate string \( s \)

\[
S = a^{P^t} \quad \text{s.t.} \quad |s| = P^t > P \checkmark
\]

Step 2: characteristics of \( uvxyz \) using \( |vy| > 0 \)

\[
vy = a^n \quad \text{for some } m > 0
\]

Step 3: finding \( i \) so that \( uv^i xy^i z \notin L \)

\[
\begin{align*}
UV^0xy^0z &= \alpha^{P^t - m} \\
UV^2xy^2z &= \alpha^{P^t - m + 2m} \\
UV^i xy^i z &= \alpha^{P^t - m + i\cdot m} = \alpha^{P^t + (i-1)m}
\end{align*}
\]

\[
\begin{align*}
(n+1)^2 &= n^2 + 2n + 1 > n^2 + 1 \\
i &= P^t s + 1
\end{align*}
\]

\[
UV^i xy^i z = \alpha^{P^t + (P^t s) m} = \alpha^{P^t (1 + s^2)}
\]

\[
UV^i xy^i z \notin L
\]
Example 2: \( \{ a^n b^m a^n : n \geq m \} \)

Step 1: finding a candidate string \( s \)

\[
S = a^p b^p a^p \quad |S| = 3p \geq p \quad \checkmark \quad S \in L \quad \checkmark
\]

Step 2: cases of \( v \) and \( y \) using \( |vxy| \leq p \)

Case 1: \( u, y \) contain an \( a \) from that 1st run of \( a \)’s

Case 2: doesn’t contain a from 1st group of \( a \)’s

\[
\rightarrow u \text{ or } y \text{ contain at least one } b
\]

\[
\rightarrow \text{ all } a \text{'s}
\]

Step 3: finding \( i \) so that \( uv^i xy^i z \notin L \)

Case 1: \( uV^{\leq i} xy^2 z \) will have more \( a \)’s at the start so \( uv^2 xy^2 z \notin L \)

Case 2:

\[
\rightarrow uV^2 xy^2 z + m \# \text{ of } b \text{'s} \Rightarrow
\]

\[
\Rightarrow uV^2 xy^2 z \notin L
\]
\[
\begin{align*}
A_{pp} & \rightarrow \epsilon \\
A_{11} & \rightarrow \epsilon \\
A_{22} & \rightarrow \epsilon \\
A_{44} & \rightarrow \epsilon \\
A_{eq} & \rightarrow A_{p^2} A_{q^2} \\
A_{12} & \rightarrow A_{12} A_{21} \\
A_{23} & \rightarrow b A_{23} \\
A_{14} & \rightarrow \epsilon A_{23} \\
A_{23} & \rightarrow b A_{23} \epsilon
\end{align*}
\]