CSCI 232:
Data Structures and Algorithms

Shortest Path (Part 1)

Reese Pearsall

Spring 2024

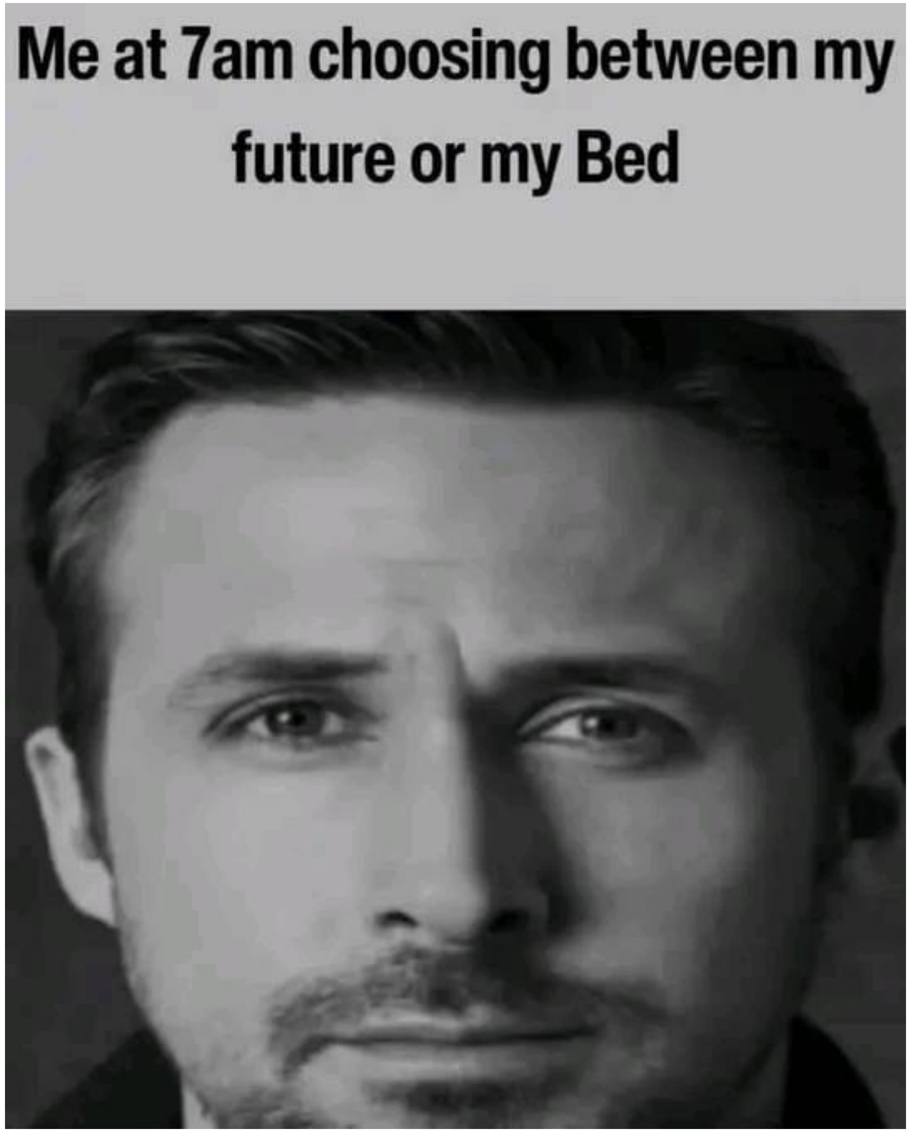
Announcements

Lab 10 due **Friday**
(Part 1 of Program 3)

No Office hours Tomorrow

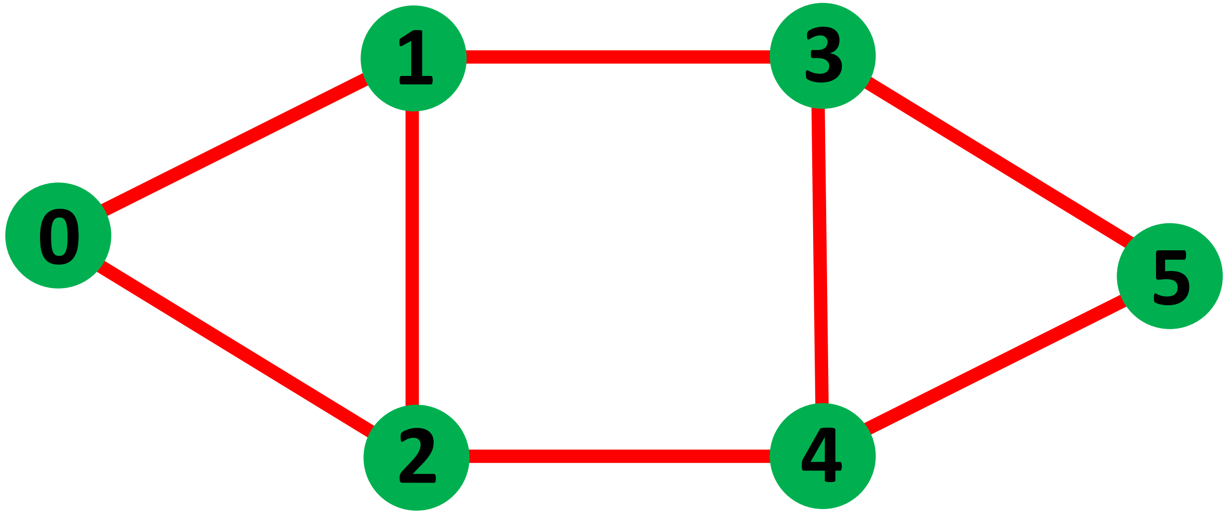
All of program 3 has been posted

Survey results



Graphs

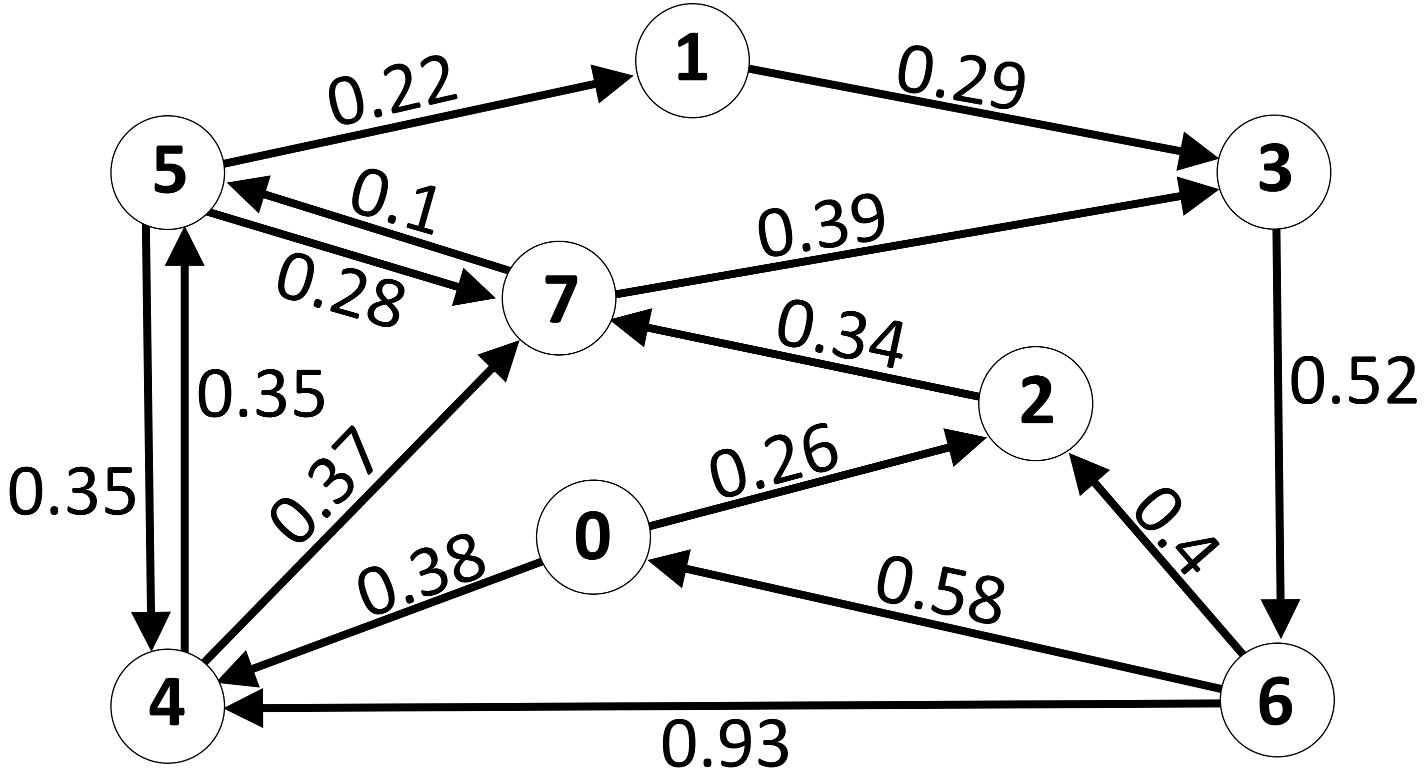
G = (V, E)



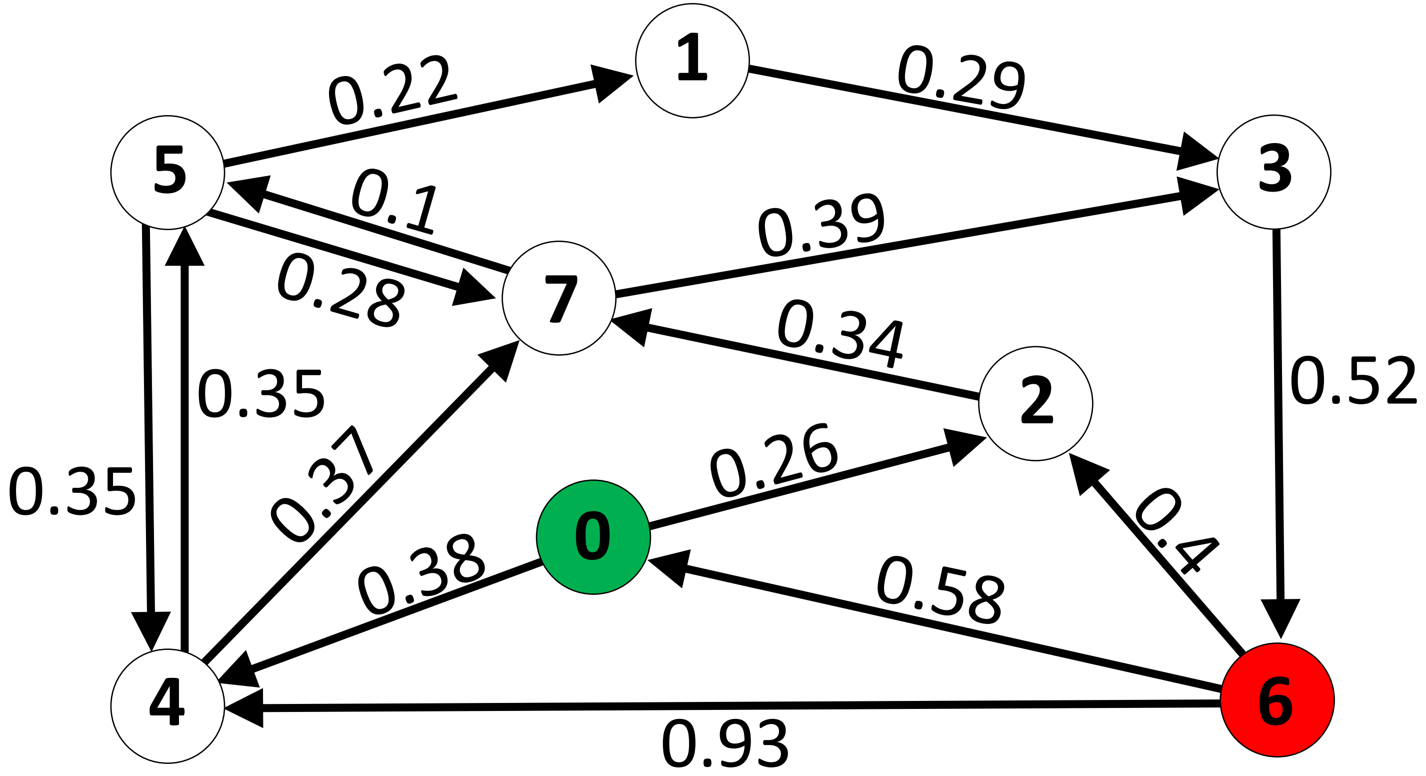
Adjacency List

0	→ {1,2}
1	→ {0,2,3}
2	→ {0,1,4}
3	→ {1,4,5}
4	→ {2,3,5}
5	→ {3,4}

Shortest Path



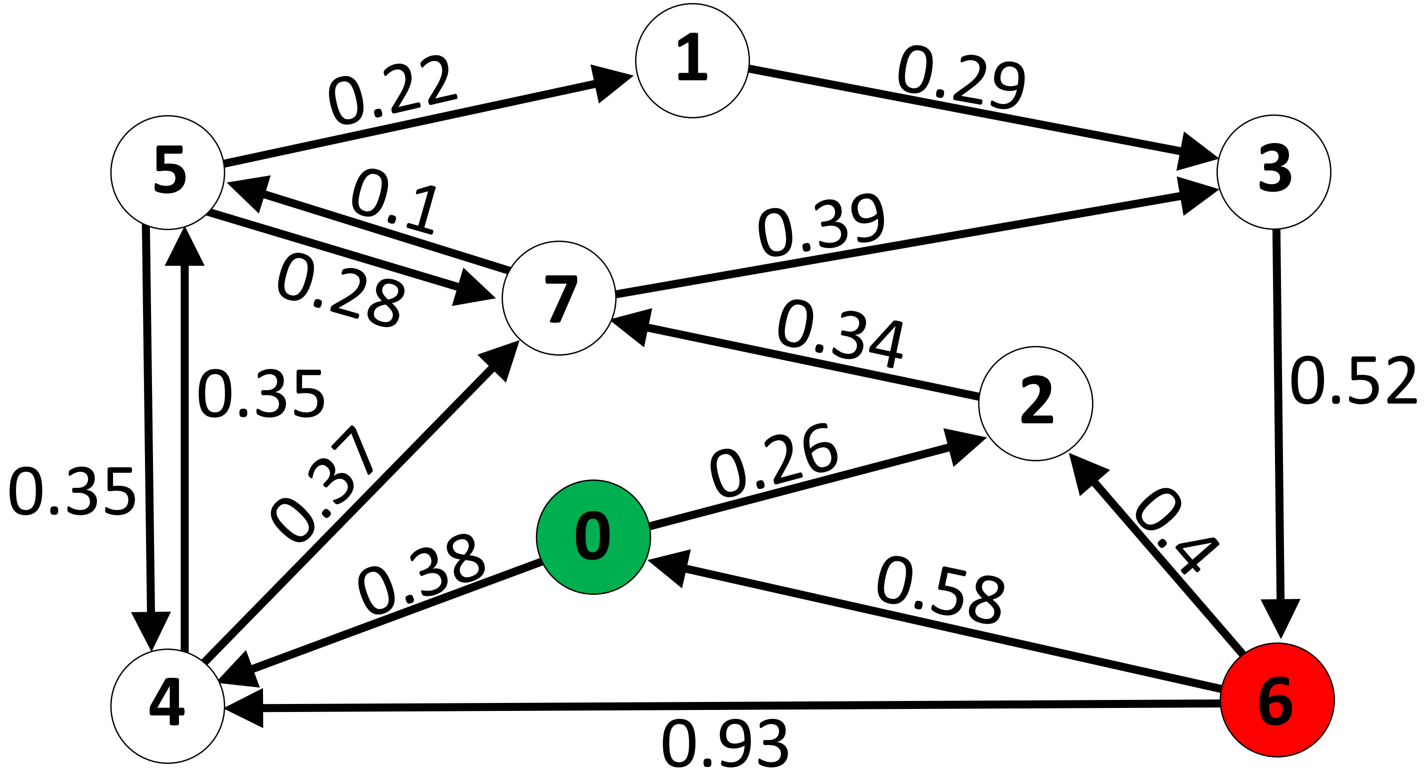
Shortest Path



Path with the smallest sum of edge weights.

What is the shortest path between **vertex 0** and **vertex 6**?

Shortest Path

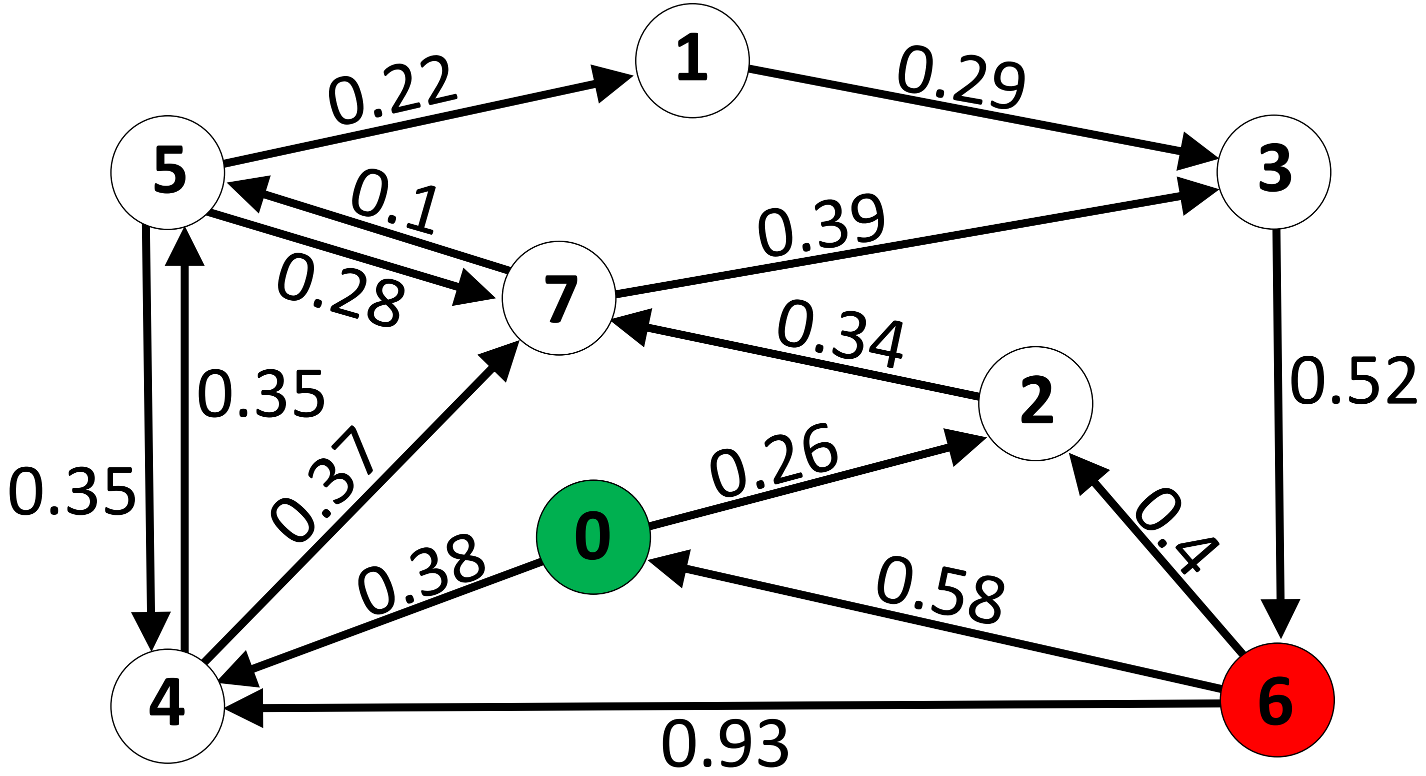


Assumptions:

- Graph is directed.
- Graph is edge-weighted.
- Edge weights are non-negative.
- Graph need not be simple (though our example will be).

What is the shortest path between vertex 0 and vertex 6?

Shortest Path



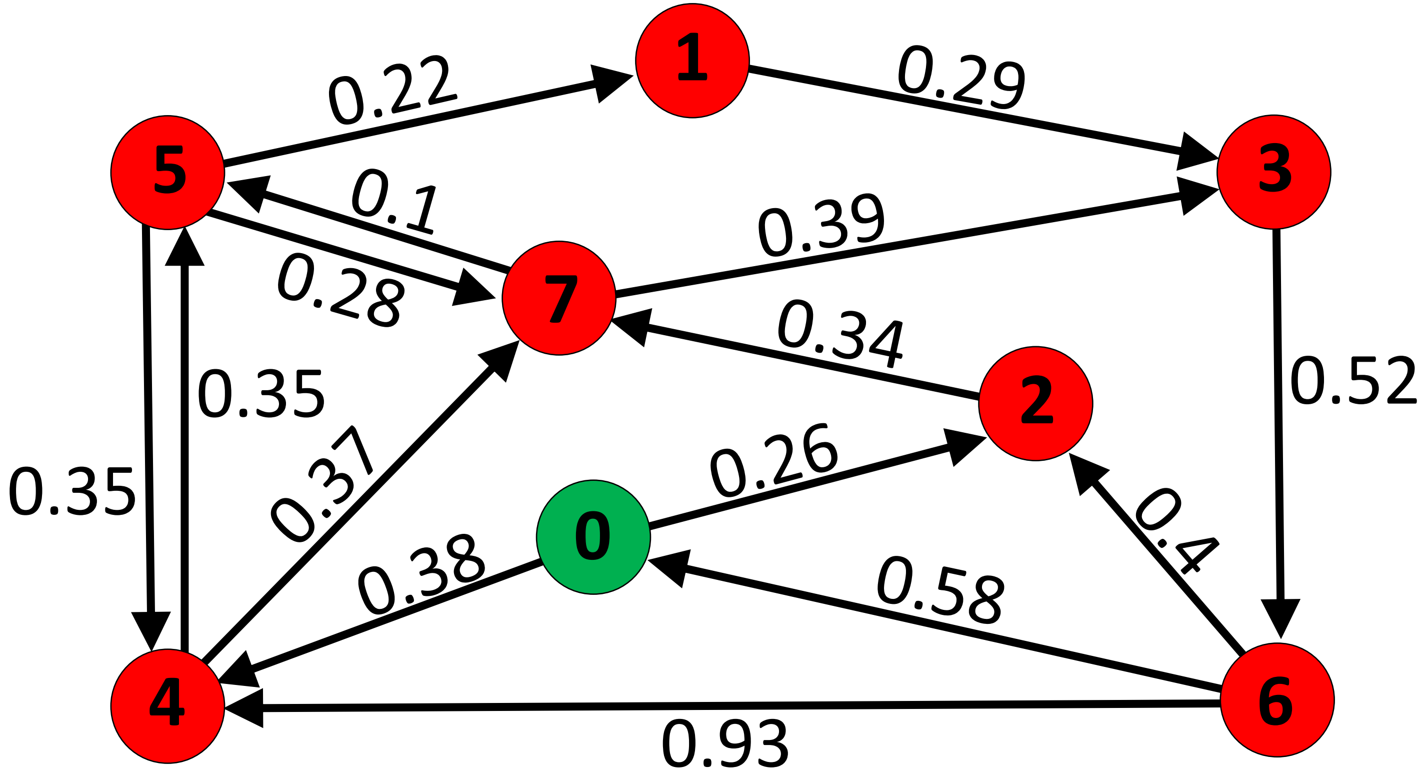
Assumptions:

- Graph is directed.
- Graph is edge-weighted.
- Edge weights are non-negative.
- Graph need not be simple (though our example will be).

Ideas?

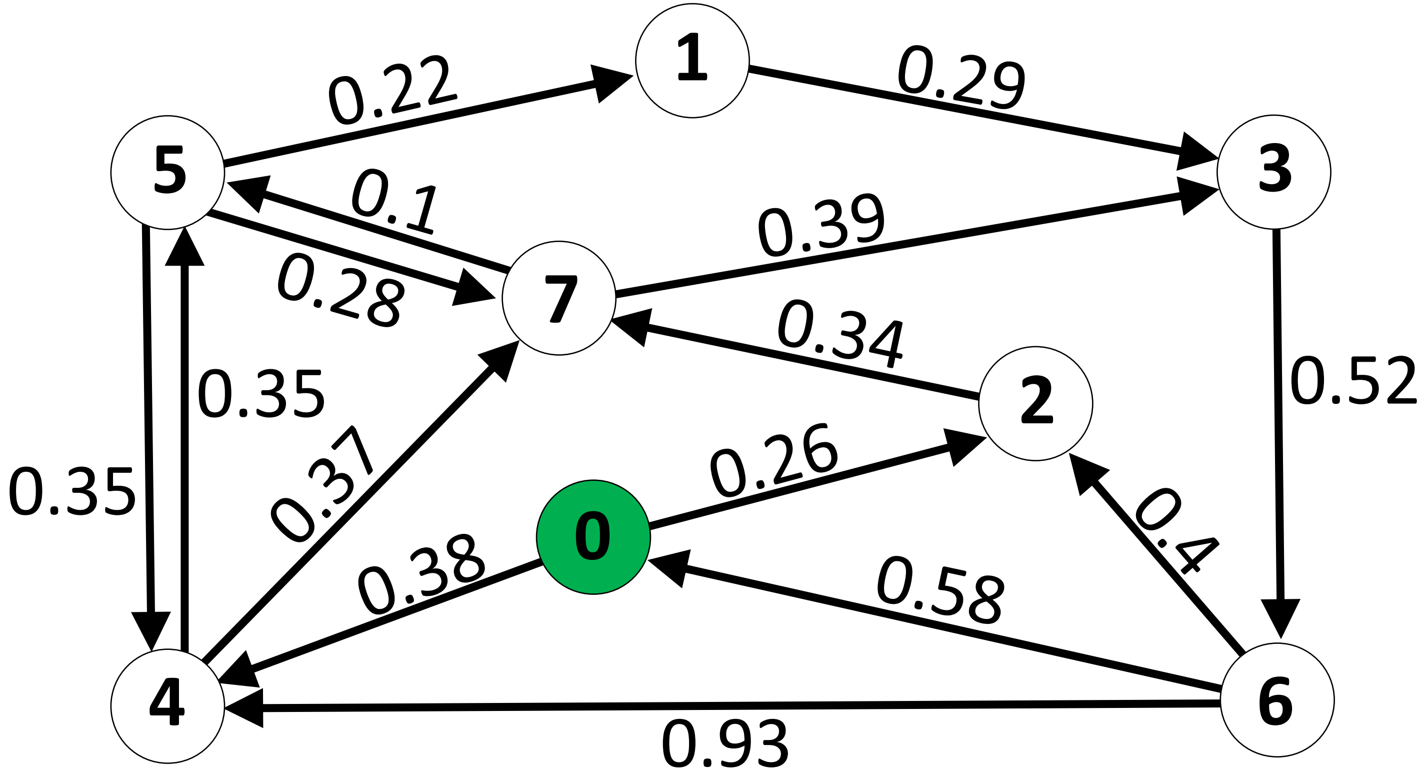
What is the shortest path between **vertex 0** and **vertex 6**?

Shortest Path



We are going to find the shortest path between vertex 0 and every other vertex, flooding out from 0.

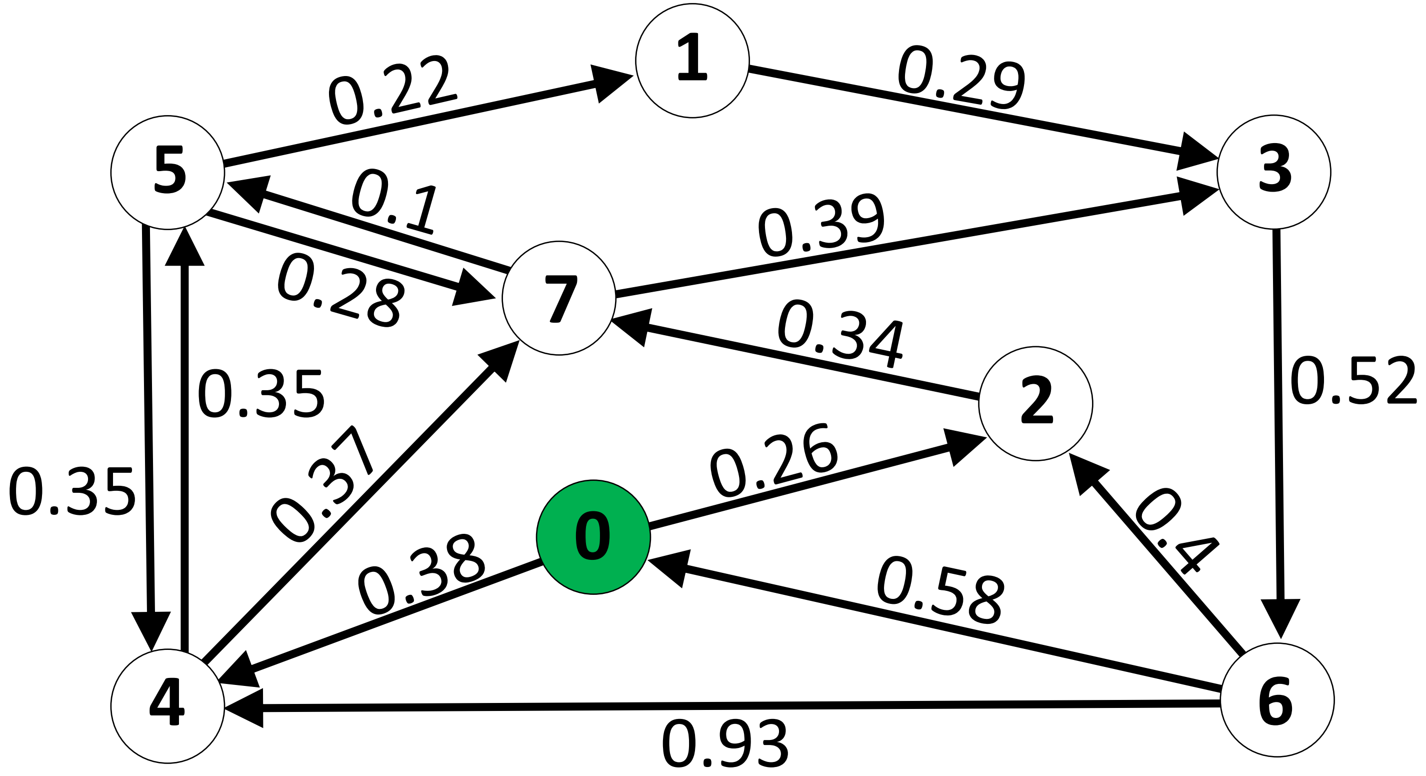
Shortest Path



Distance from 0

0	?
1	?
2	?
3	?
4	?
5	?
6	?
7	?

Shortest Path



Distance from 0

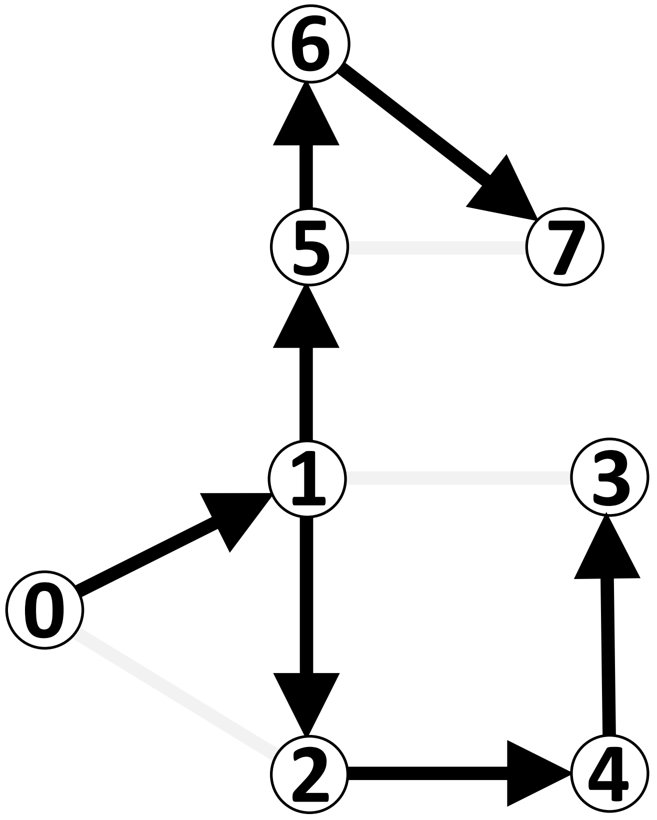
0	0
1	∞
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞

How can we keep track of routes?

Graphs - Paths

`int[] previousVertex`

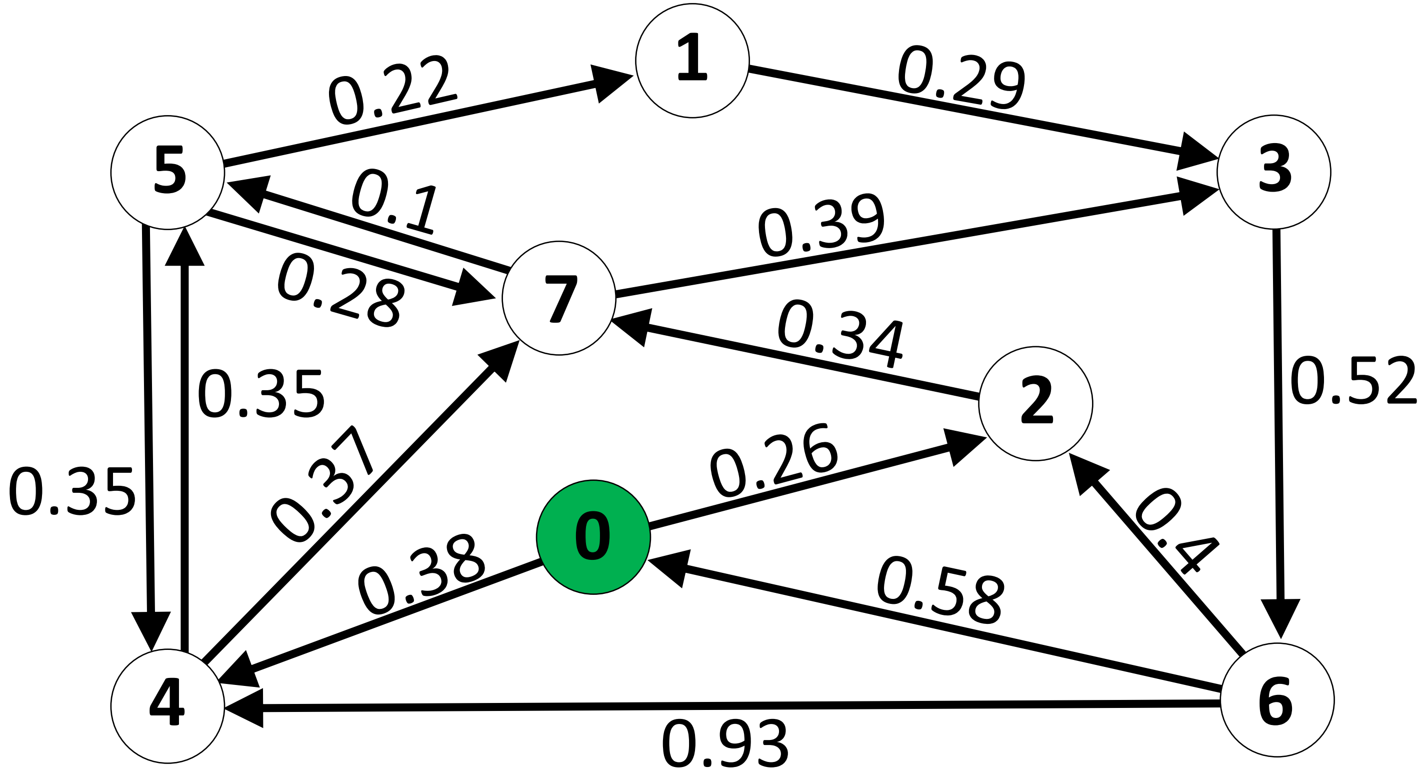
0	-
1	0
2	1
3	4
4	2
5	1
6	5
7	6



How do we determine the path from 0 to 6?

Start at vertex 6. Find its previous vertex. Find its previous vertex… until we get back to the start (0).

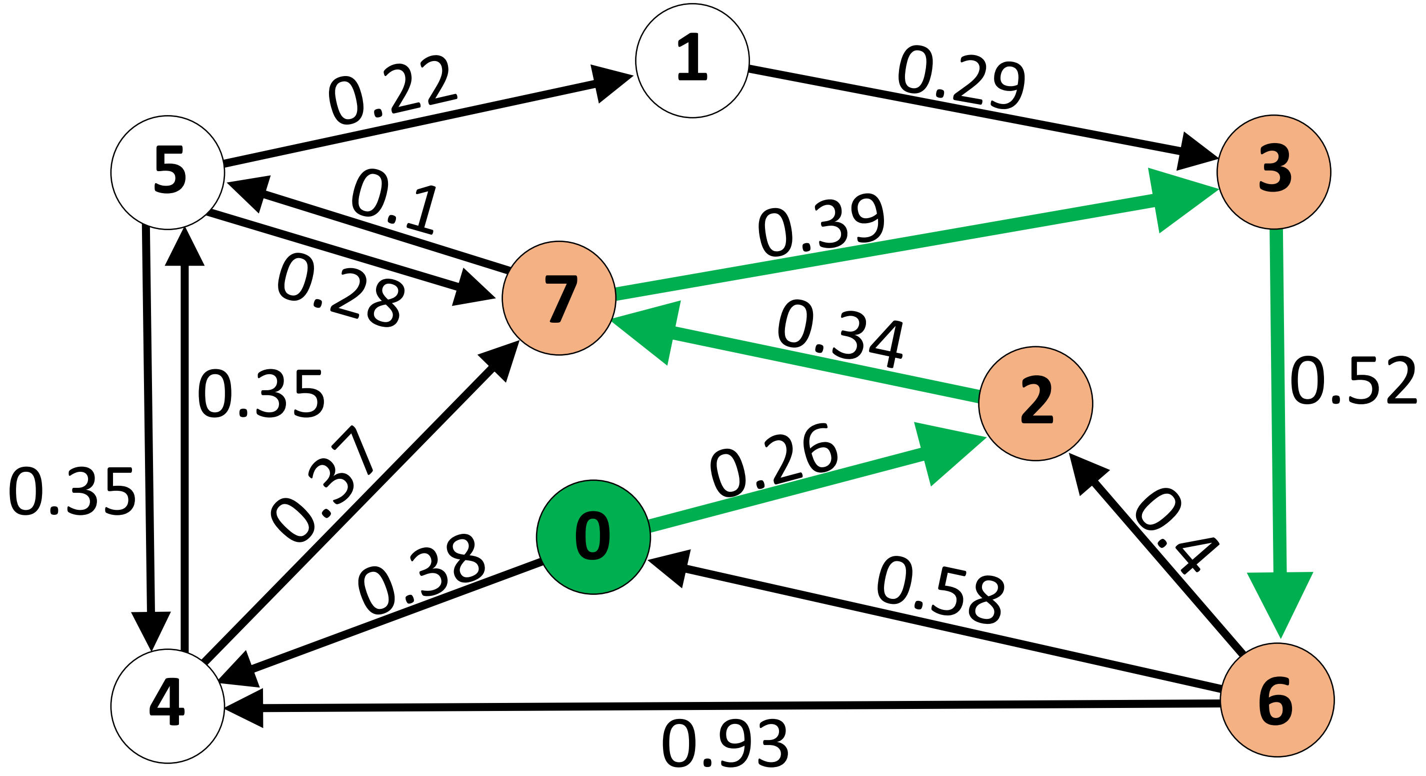
Shortest Path



	Distance from 0	Previous vertex
0	0	-
1	∞	
2	∞	
3	∞	
4	∞	
5	∞	
6	∞	
7	∞	

How can we keep track of routes?

Shortest Path

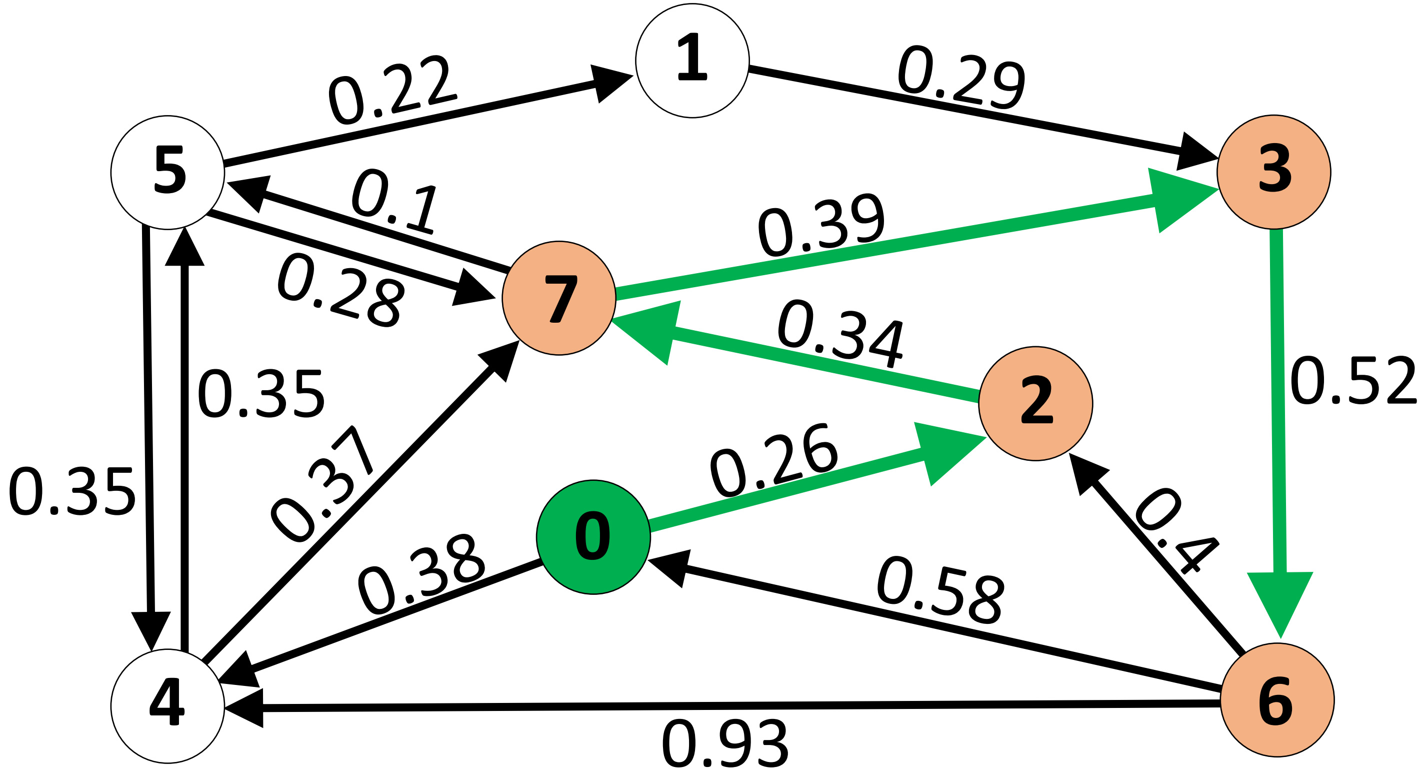


	Distance from 0
0	0
1	∞
2	0.26
3	0.99
4	∞
5	∞
6	1.51
7	0.60

	Previous vertex
0	-
1	
2	0
3	7
4	
5	
6	3
7	2

How can we keep track of routes?

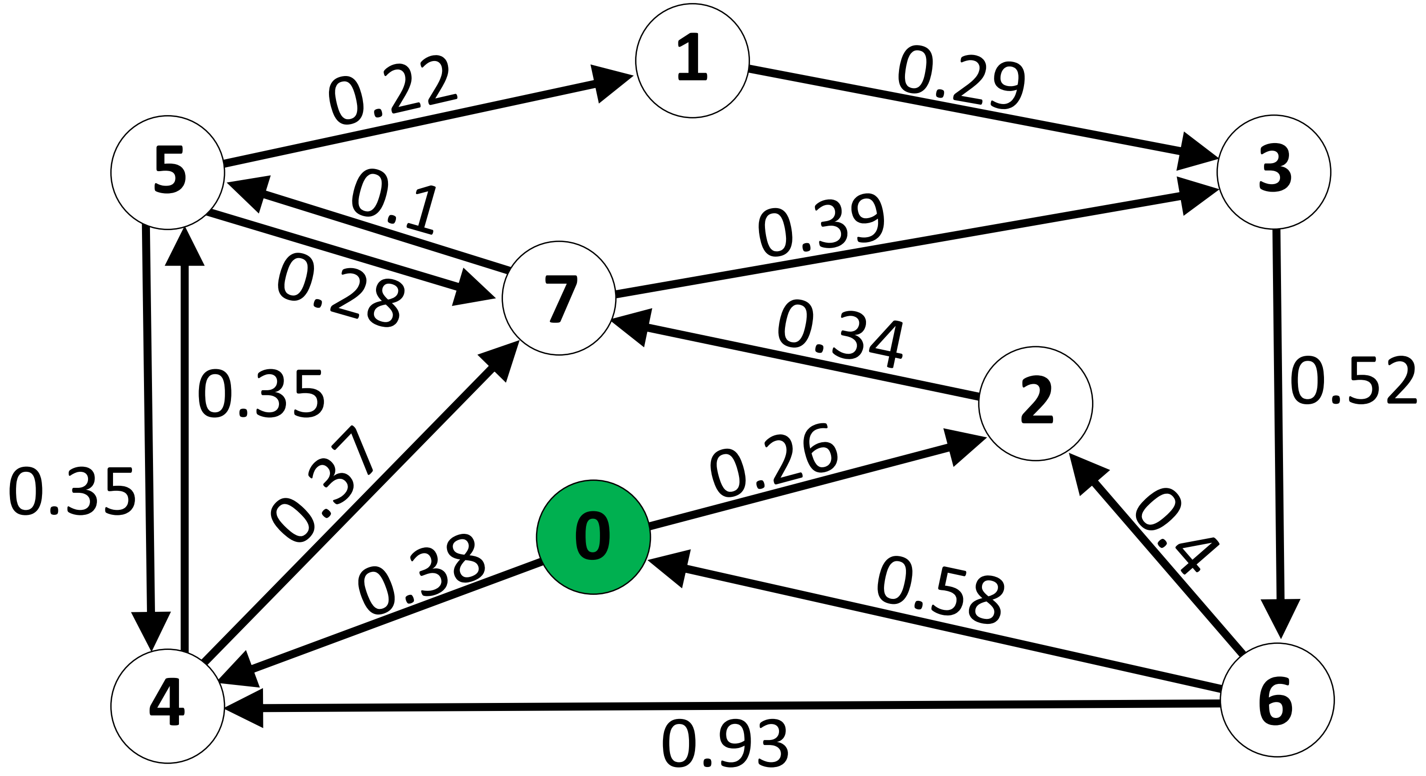
Shortest Path



	Distance from 0	Previous vertex
0	0	-
1	∞	
2	∞	0
3	∞	7
4	∞	
5	∞	
6	∞	3
7	∞	2

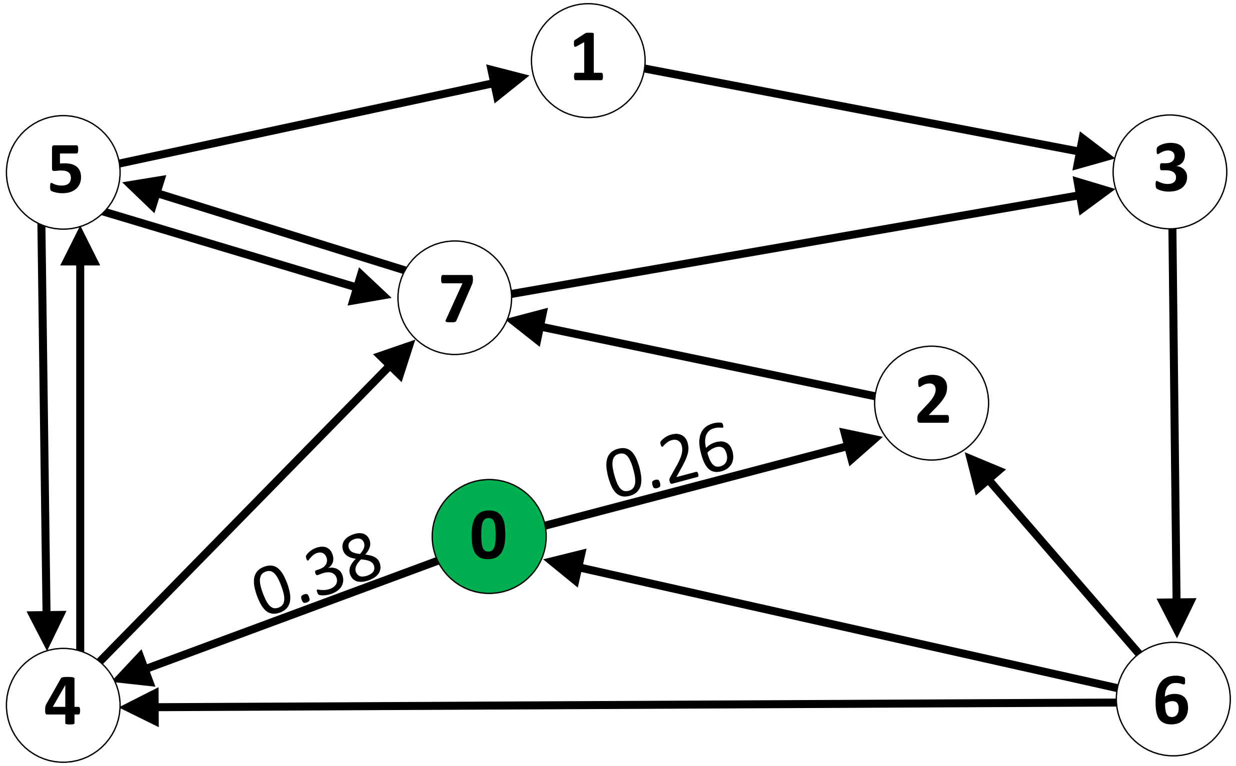
If this is the shortest path from 0 to 6, what can we say about the shortest path from 0 to 3?

Shortest Path



	Distance from 0	Previous vertex
0	0	-
1	∞	
2	∞	
3	∞	
4	∞	
5	∞	
6	∞	
7	∞	

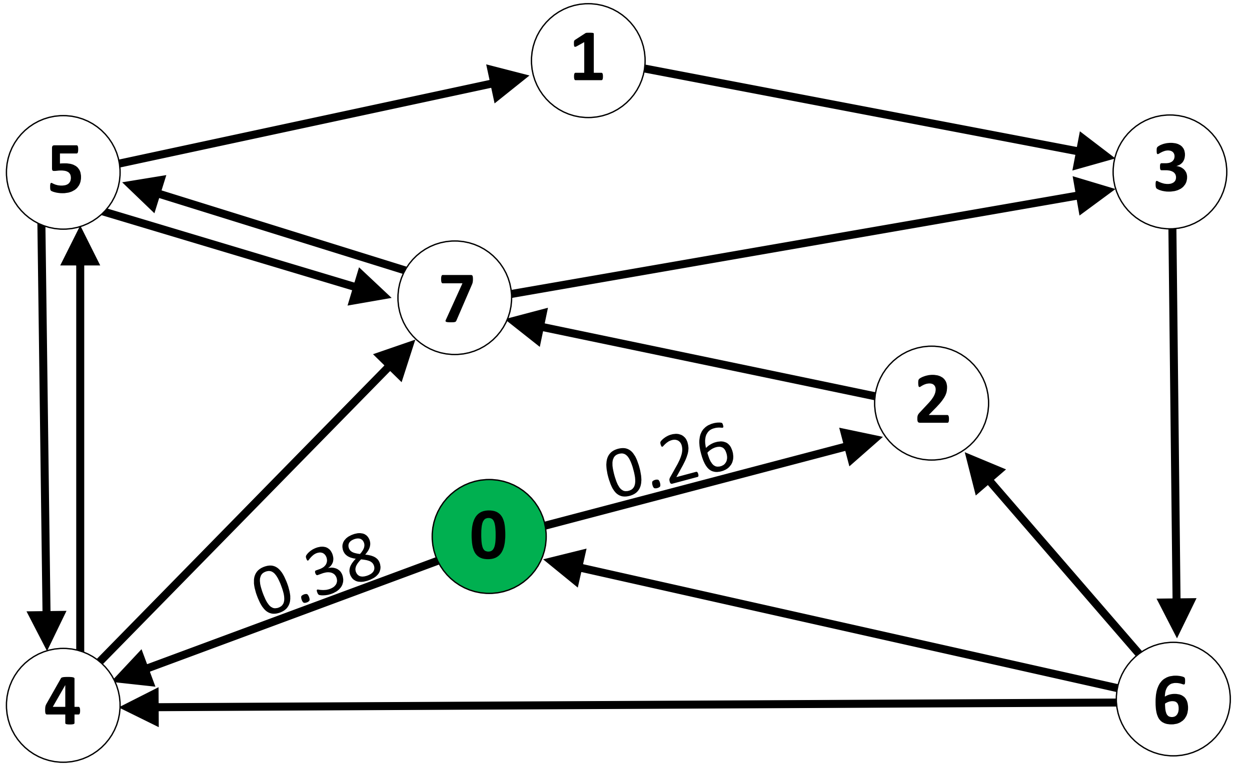
Shortest Path



	Distance from 0	Previous vertex
0	0	-
1	∞	
2	∞	
3	∞	
4	∞	
5	∞	
6	∞	
7	∞	

Claim: There cannot possibly be a shorter path from 0 to 2 than the edge from 0 to 2 because…?

Shortest Path

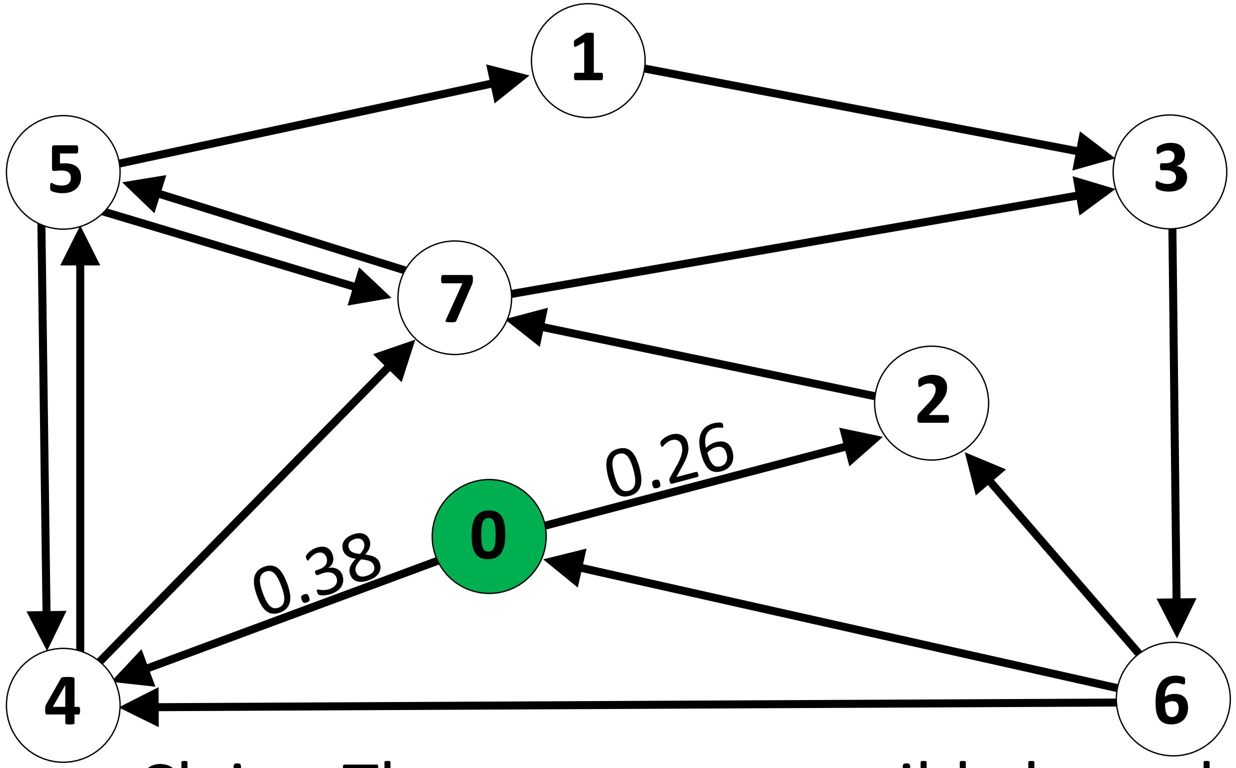


	Distance from 0	Previous vertex
0	0	-
1	∞	
2	∞	
3	∞	
4	∞	
5	∞	
6	∞	
7	∞	

Claim: There cannot possibly be a shorter path from 0 to 2 than the edge from 0 to 2 because non-negative edge weights mean every other path is at least 0.38 or 0.26.

Shortest Path

	Distance from 0	Previous vertex
0	0	-
1	∞	
7	∞	

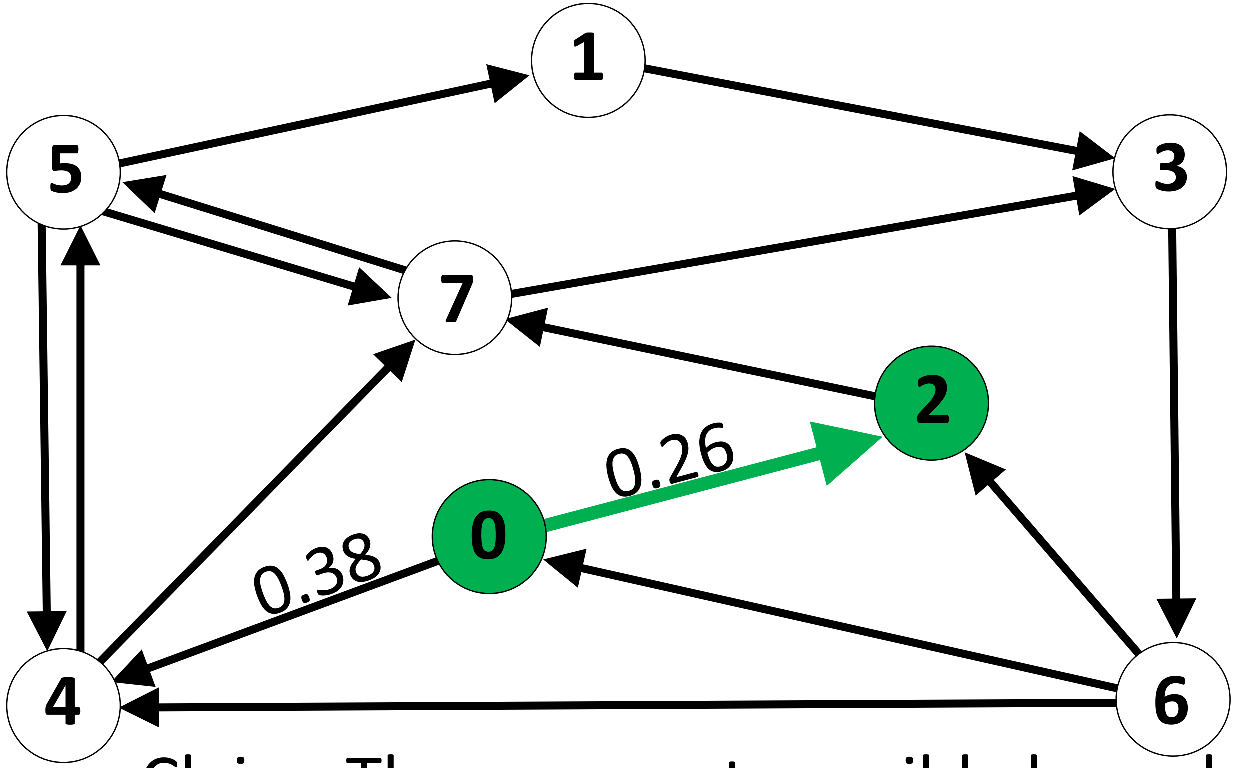


Can we say the same thing about the edge from 0 to 4?

I.e., Could there be a shorter path from 0 to 4 other than the edge from 0 to 4?

Claim: There cannot possibly be a shorter path from 0 to 2 than the edge from 0 to 2 because non-negative edge weights mean every other path is at least 0.38 or 0.26.

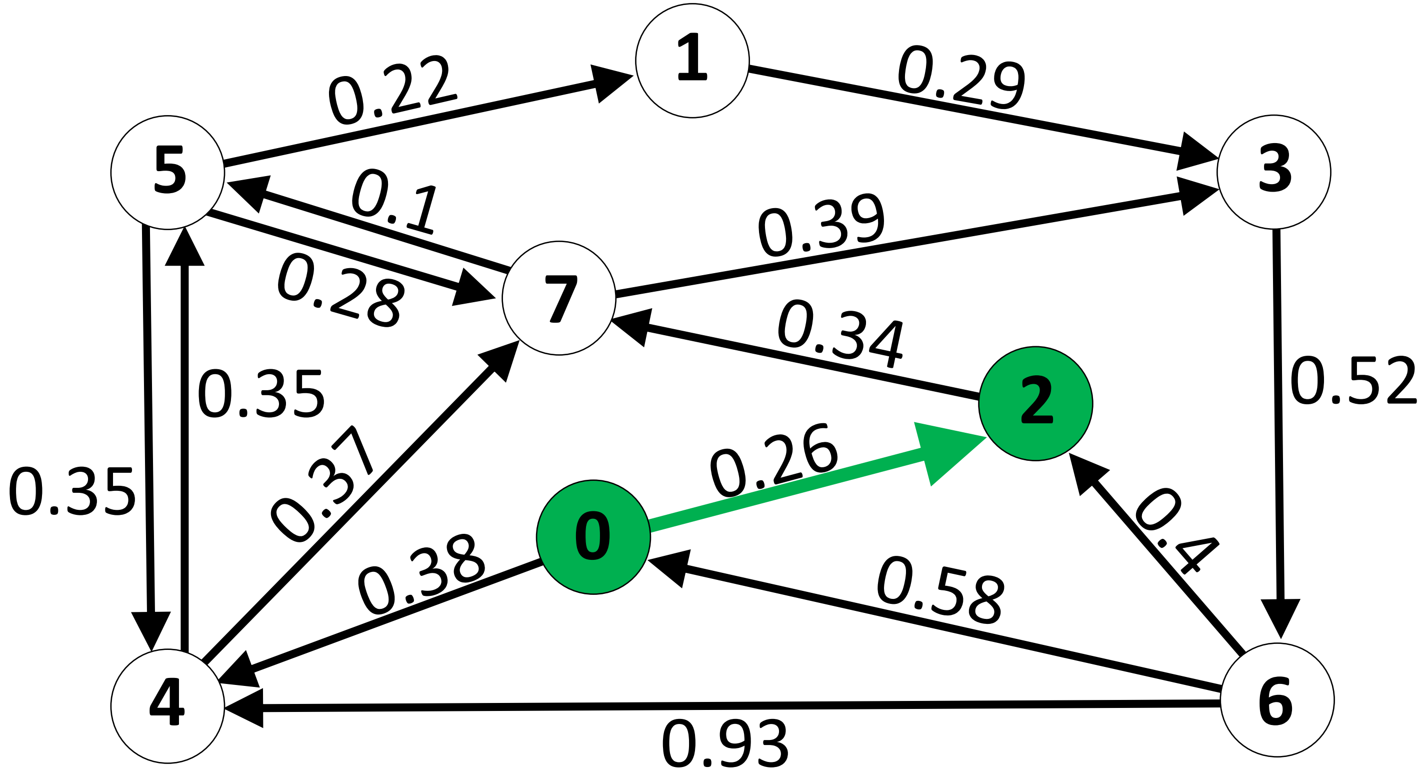
Shortest Path



	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	∞	
5	∞	
6	∞	
7	∞	

Claim: There cannot possibly be a shorter path from 0 to 2 than the edge from 0 to 2 because non-negative edge weights mean every other path is at least 0.38 or 0.26.

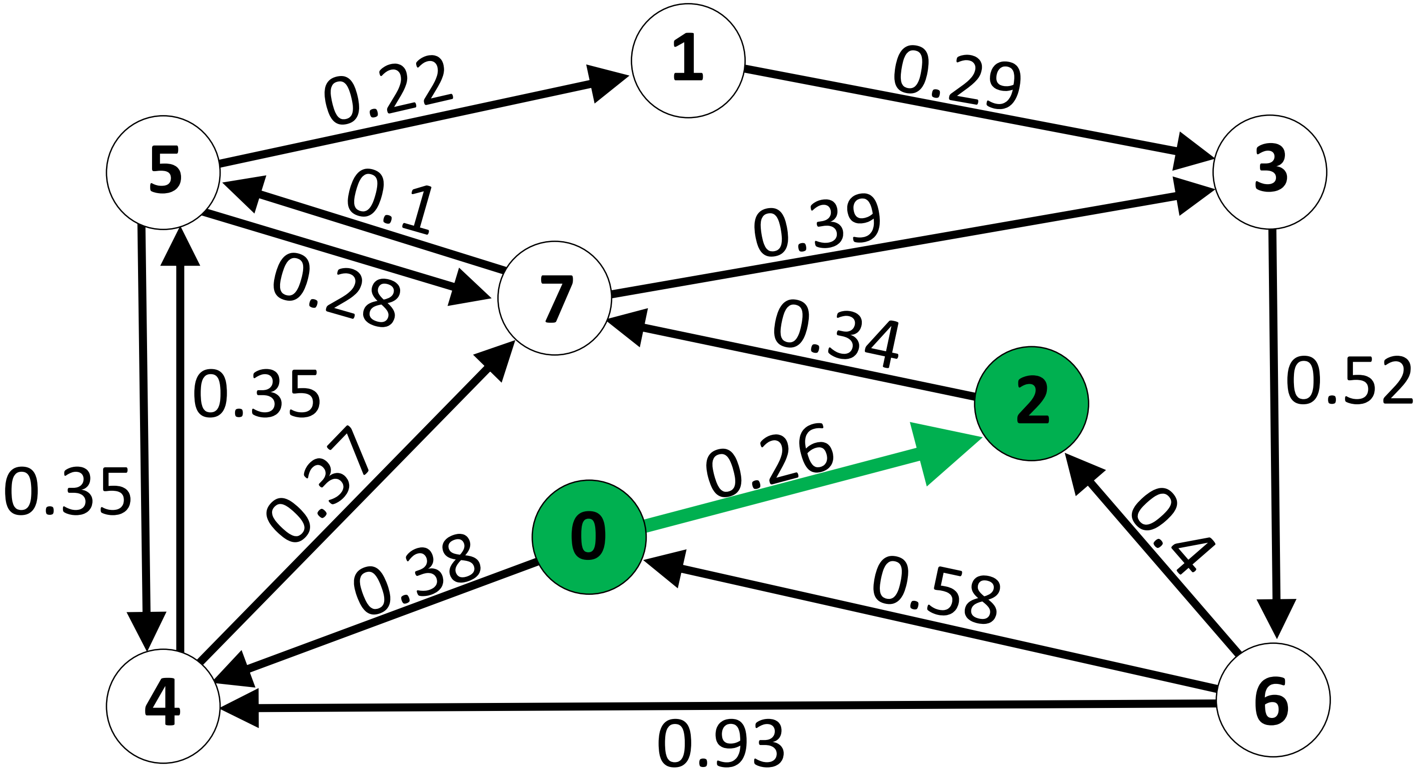
Shortest Path



	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	∞	
5	∞	
6	∞	
7	∞	

We need some process for progressing through the graph.

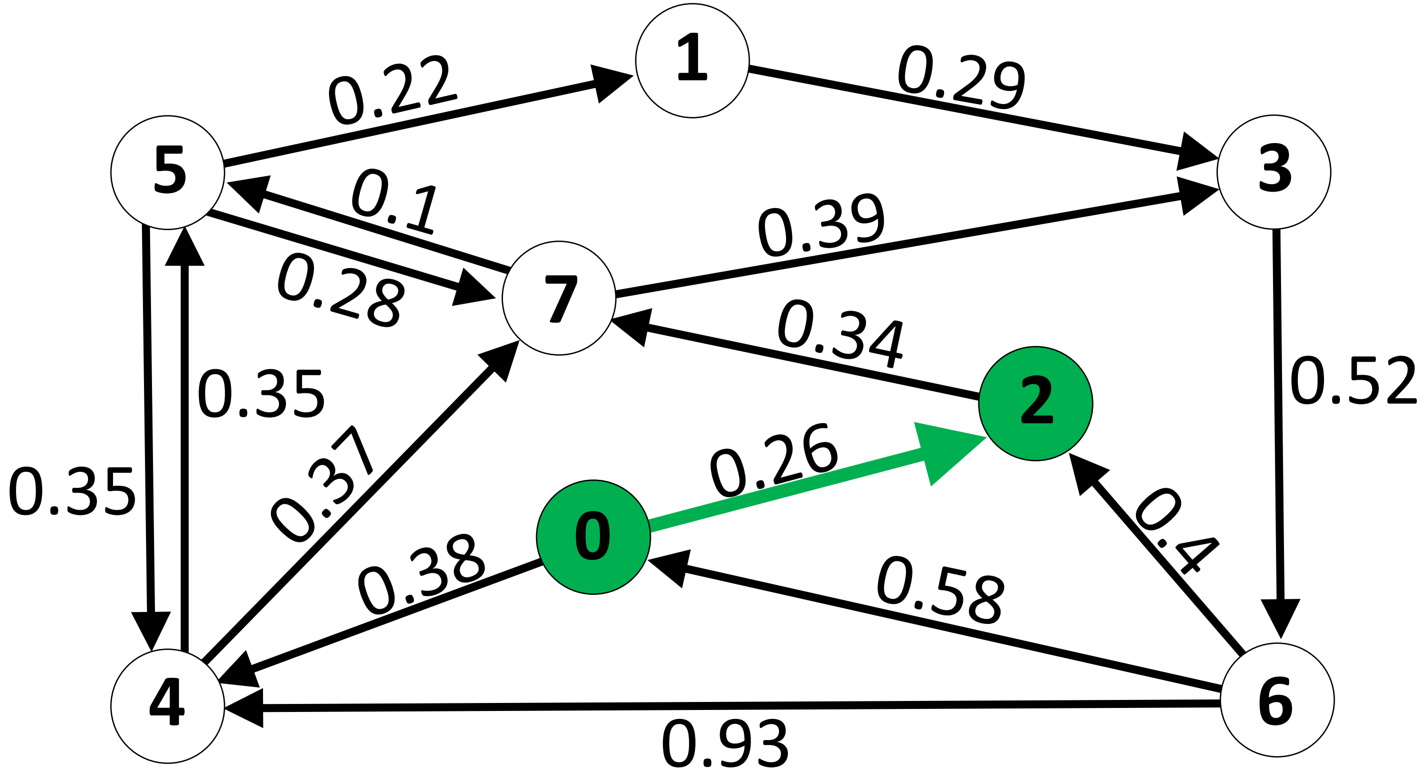
Shortest Path



	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	∞	
5	∞	
6	∞	
7	∞	

We need some process for progressing through the graph.
What if we prioritized neighbors based on path (not edge) distance?

Shortest Path



	Distance from 0
0	0
1	∞
2	0.26
3	∞
4	∞
5	∞
6	∞
7	∞

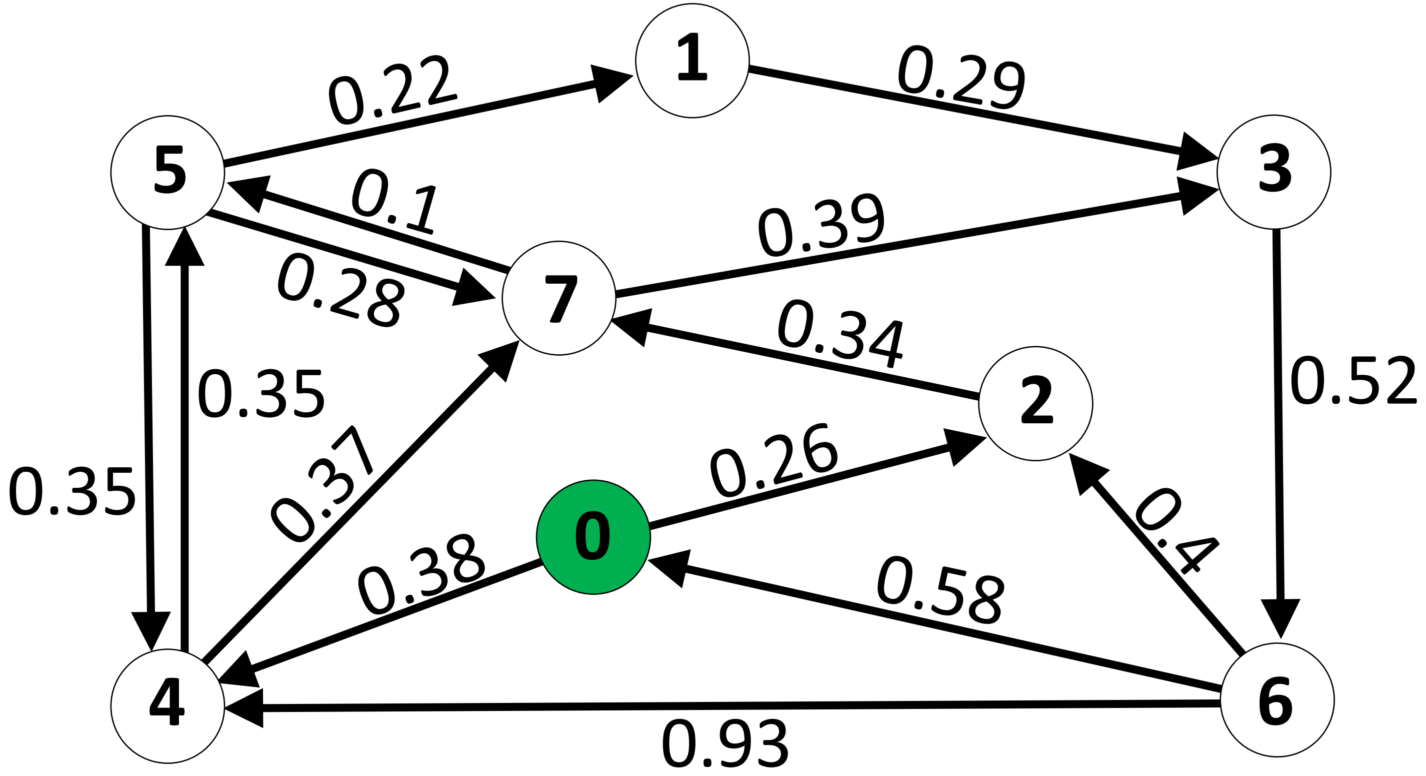
	Previous vertex
0	-
1	
2	0
3	
4	
5	
6	
7	

Priority queue

We need some process for progressing through the graph.
What if we prioritized neighbors based on path (not edge) distance?

vertex (distance)

Shortest Path



Distance from 0

0	0
1	∞
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞

Previous vertex

0	-
1	
2	
3	
4	
5	
6	
7	

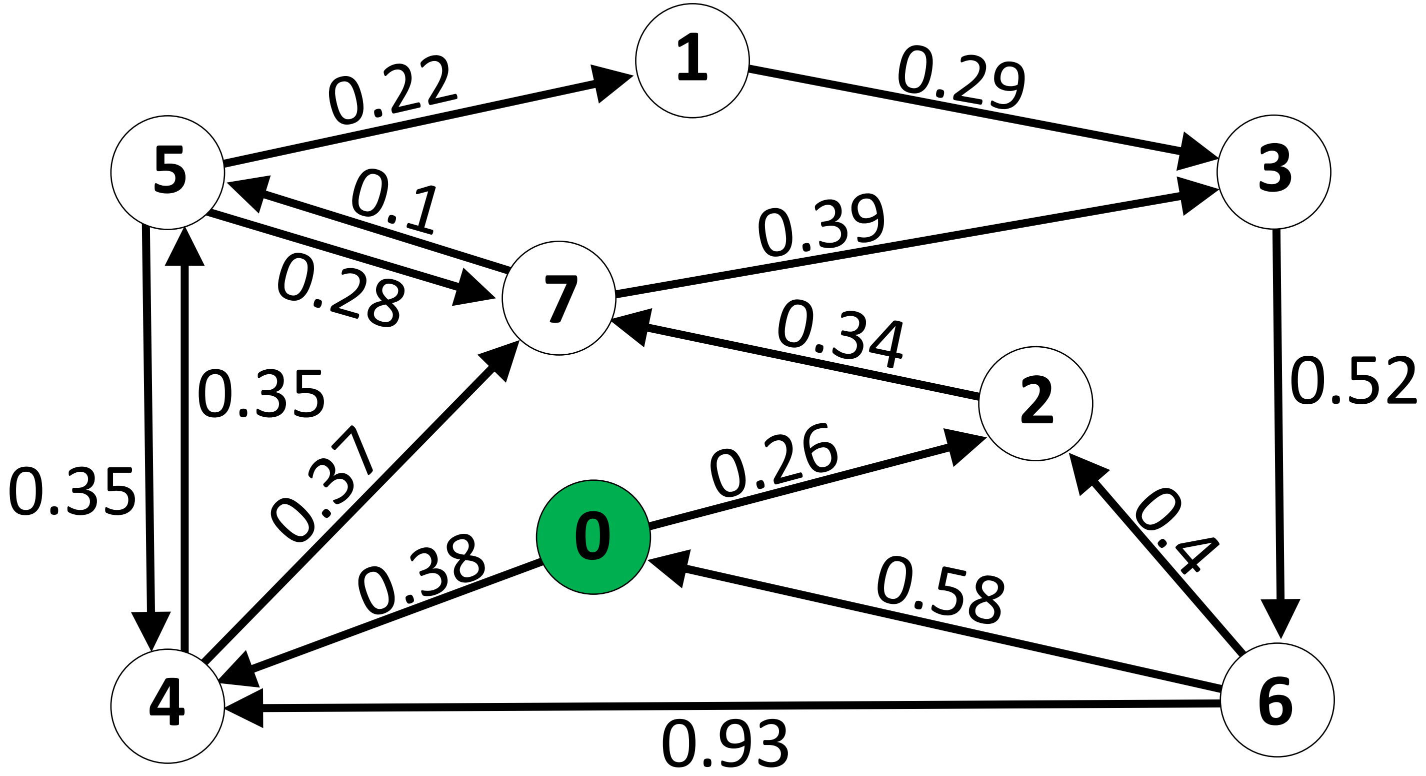
Priority queue

We need some process for progressing through the graph.

What if we prioritized neighbors based on path (not edge) distance?

vertex (distance)

Shortest Path



	Distance from 0
0	0
1	∞
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞

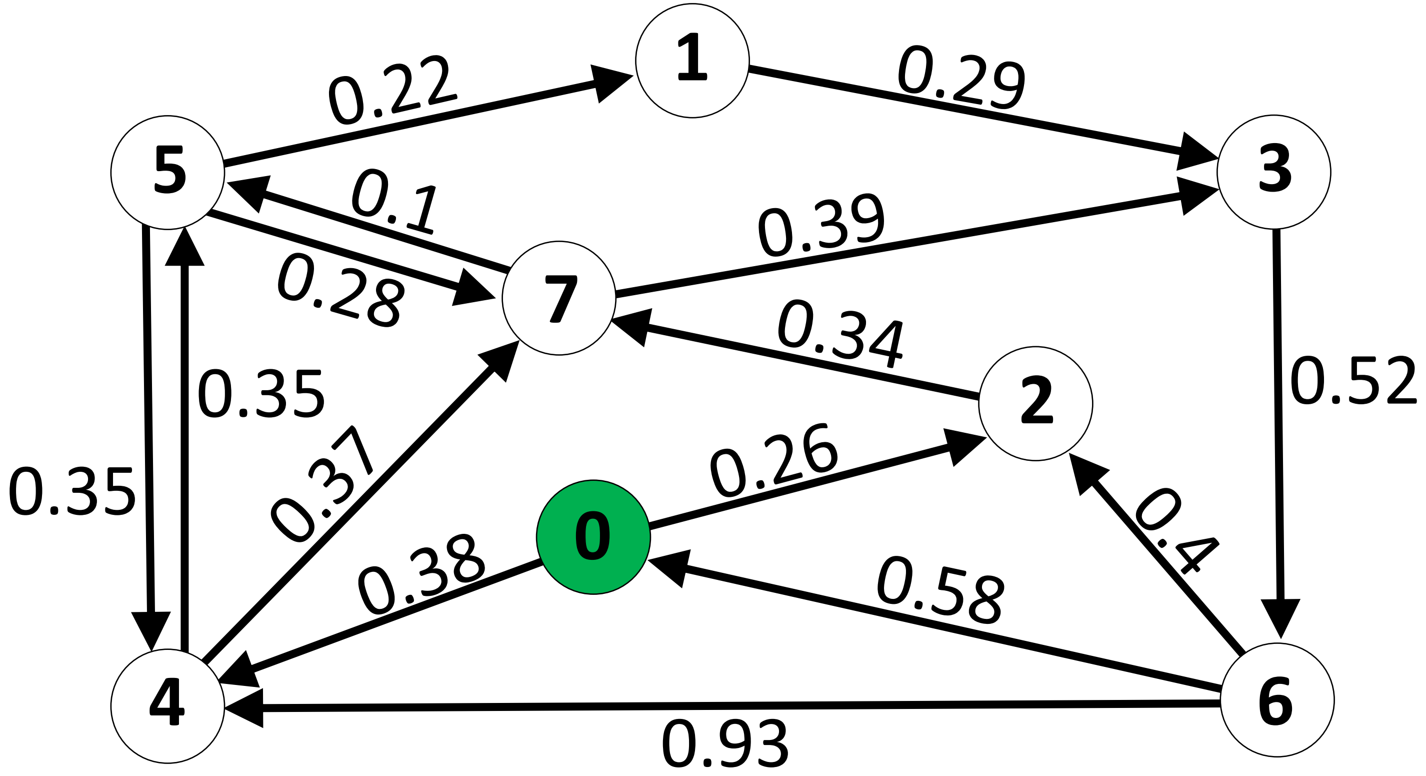
	Previous vertex
0	-
1	
2	
3	
4	
5	
6	
7	

Priority queue

vertex (distance)

What can we reach from connected vertices and at what distance (from 0)?

Shortest Path



	Distance from 0
0	0
1	∞
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞

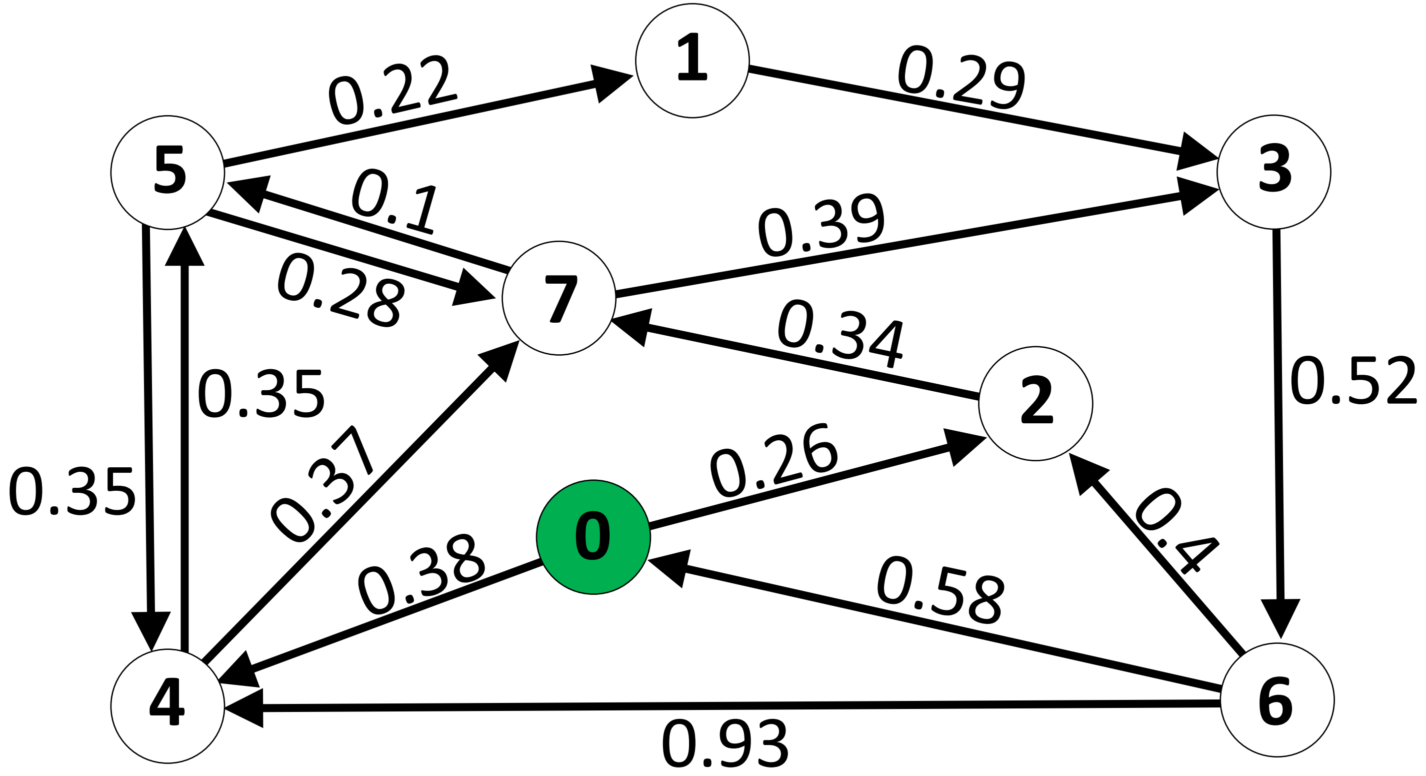
	Previous vertex
0	-
1	
2	
3	
4	
5	
6	
7	

Priority queue
2 (0.26)
4 (0.38)

vertex (distance)

What can we reach from connected vertices and at what distance (from 0)?

Shortest Path



Vertex	Distance from 0	Previous vertex
0	0	-
1	∞	
2	∞	0
3	∞	
4	∞	0
5	∞	
6	∞	
7	∞	

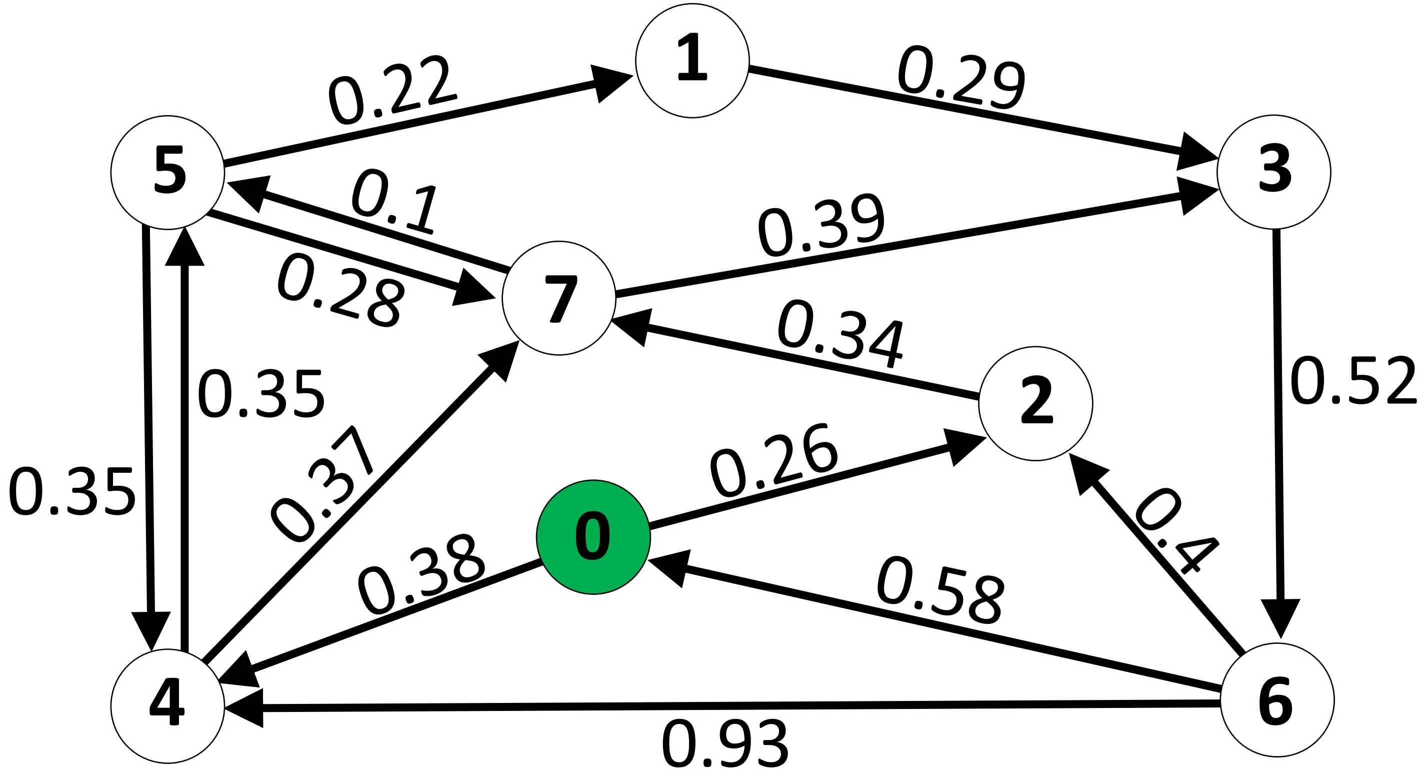
Priority queue

2 (0.26)
4 (0.38)

vertex (distance)

What can we reach from connected vertices and at what distance (from 0)?

Shortest Path



	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	0.38	0
5	∞	
6	∞	
7	∞	

Priority queue

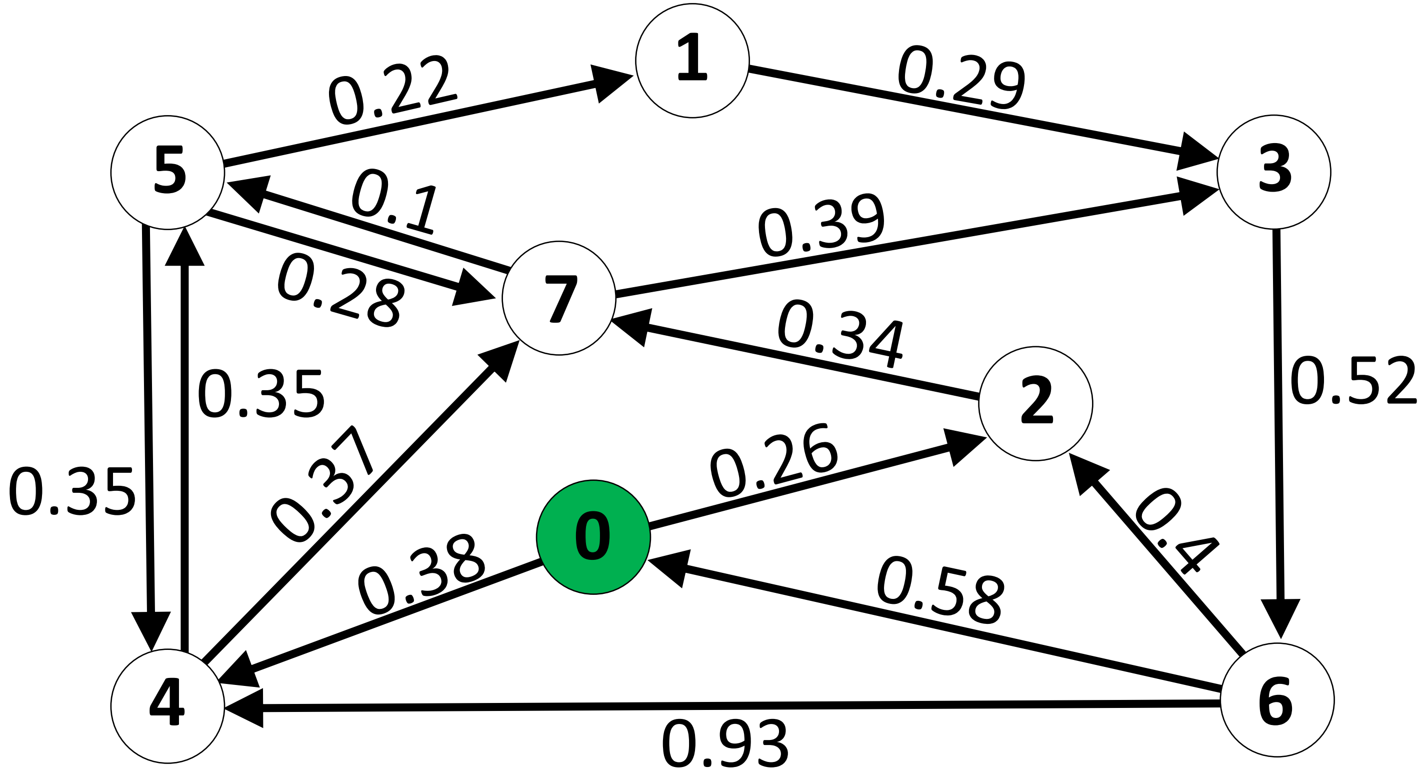
2 (0.26)
4 (0.38)

vertex (distance)

What can we reach from connected vertices and at what distance (from 0)?

Shortest Path

queue top = 2 (0.26)



	Distance from 0
0	0
1	∞
2	0.26
3	∞
4	0.38
5	∞
6	∞
7	∞

	Previous vertex
0	-
1	
2	0
3	
4	0
5	
6	
7	

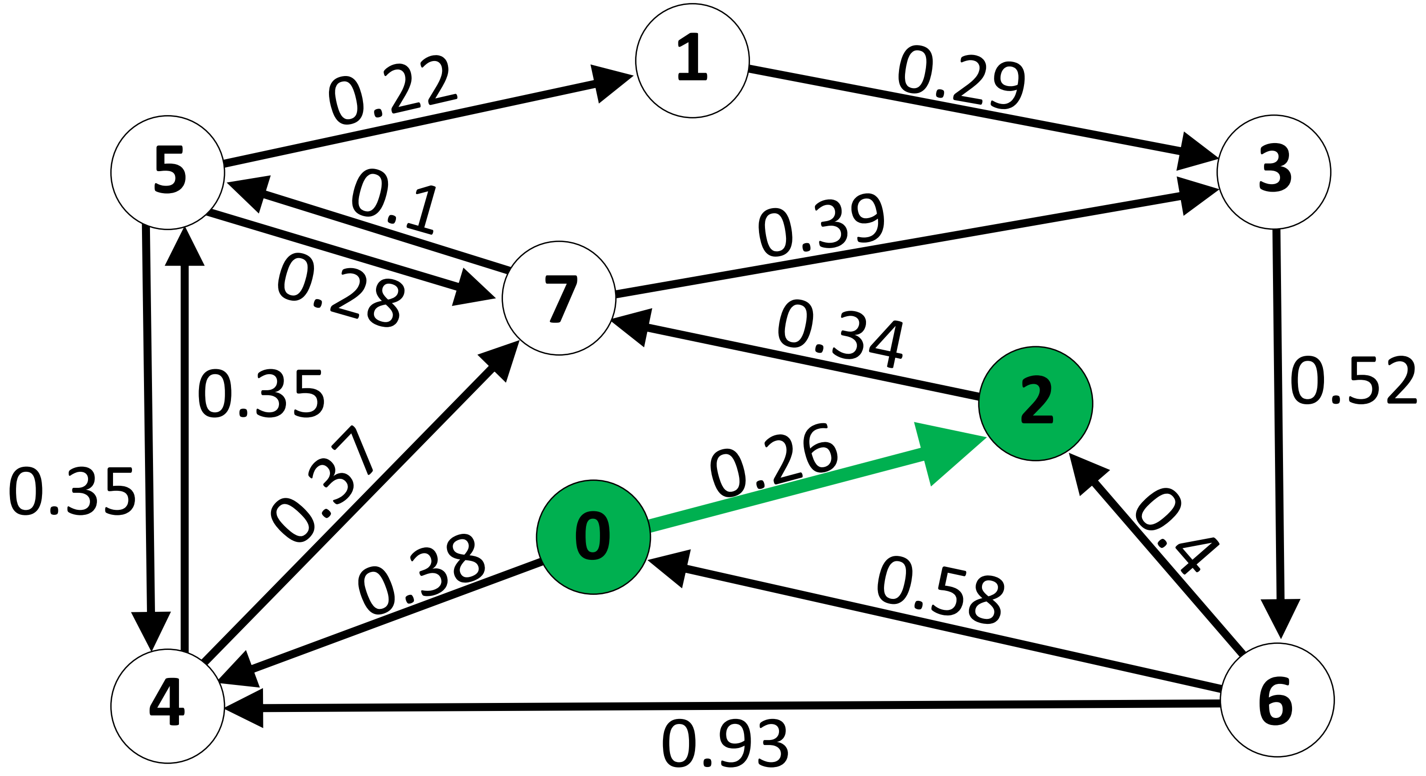
Priority queue
4 (0.38)

vertex (distance)

What can we reach from connected vertices and at what distance (from 0)?

Shortest Path

queue top = 2 (0.26)



	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	0.38	0
5	∞	
6	∞	
7	∞	

Priority queue

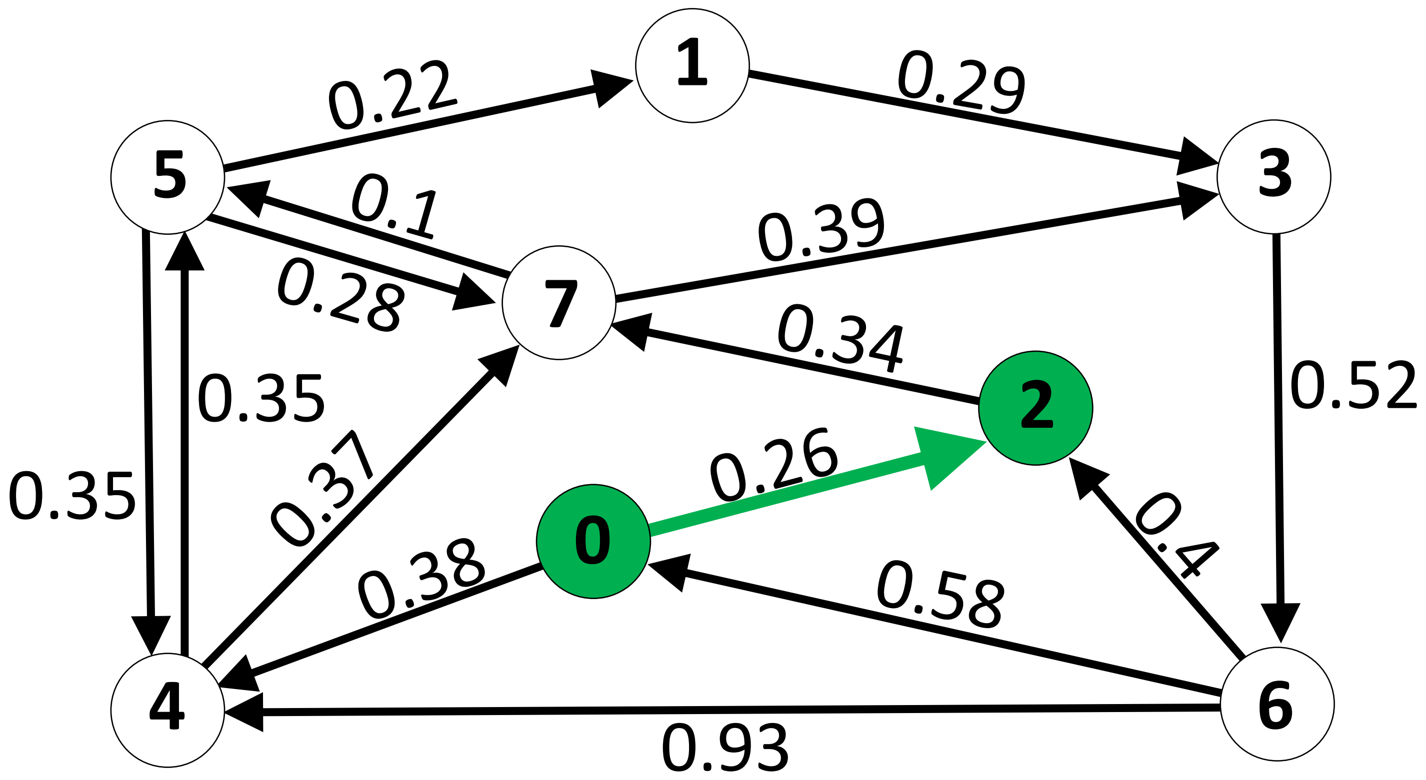
4 (0.38)

vertex (distance)

What can we reach from connected vertices and at what distance (from 0)?

Shortest Path

queue top = 2 (0.26)



	Distance from 0
0	0
1	∞
2	0.26
3	∞
4	0.38
5	∞
6	∞
7	0.60

	Previous vertex
0	-
1	
2	0
3	
4	0
5	
6	
7	2

Priority queue

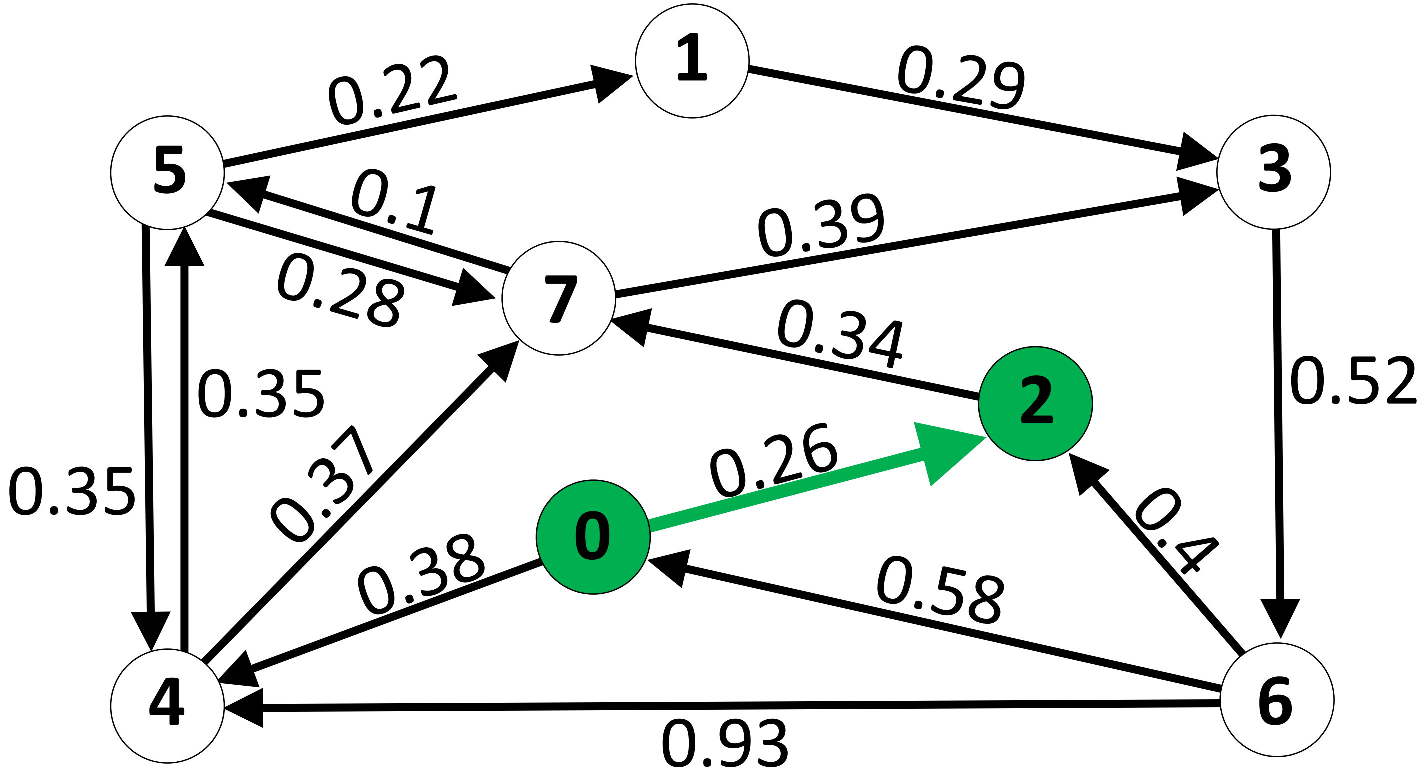
4 (0.38)
7 (0.60)

vertex (distance)

What can we reach from connected vertices and at what distance (from 0)?

Shortest Path

queue top	= 4 (0.38)



Repeat.

Vertex	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	0.38	0
5	∞	
6	∞	
7	0.60	2

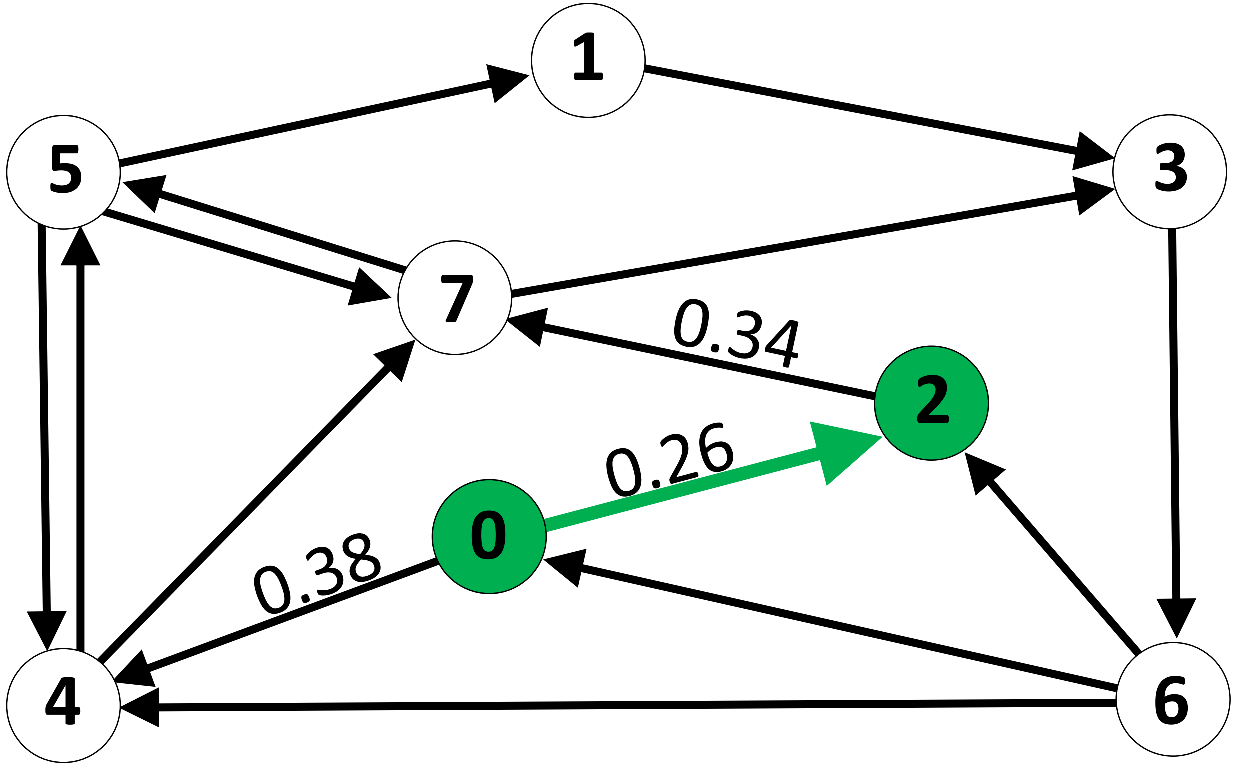
Priority queue

7 (0.60)

vertex (distance)

Shortest Path

queue top = 4 (0.38)



Vertex	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	0.38	0
5	∞	
6	∞	
7	0.60	2

Priority queue

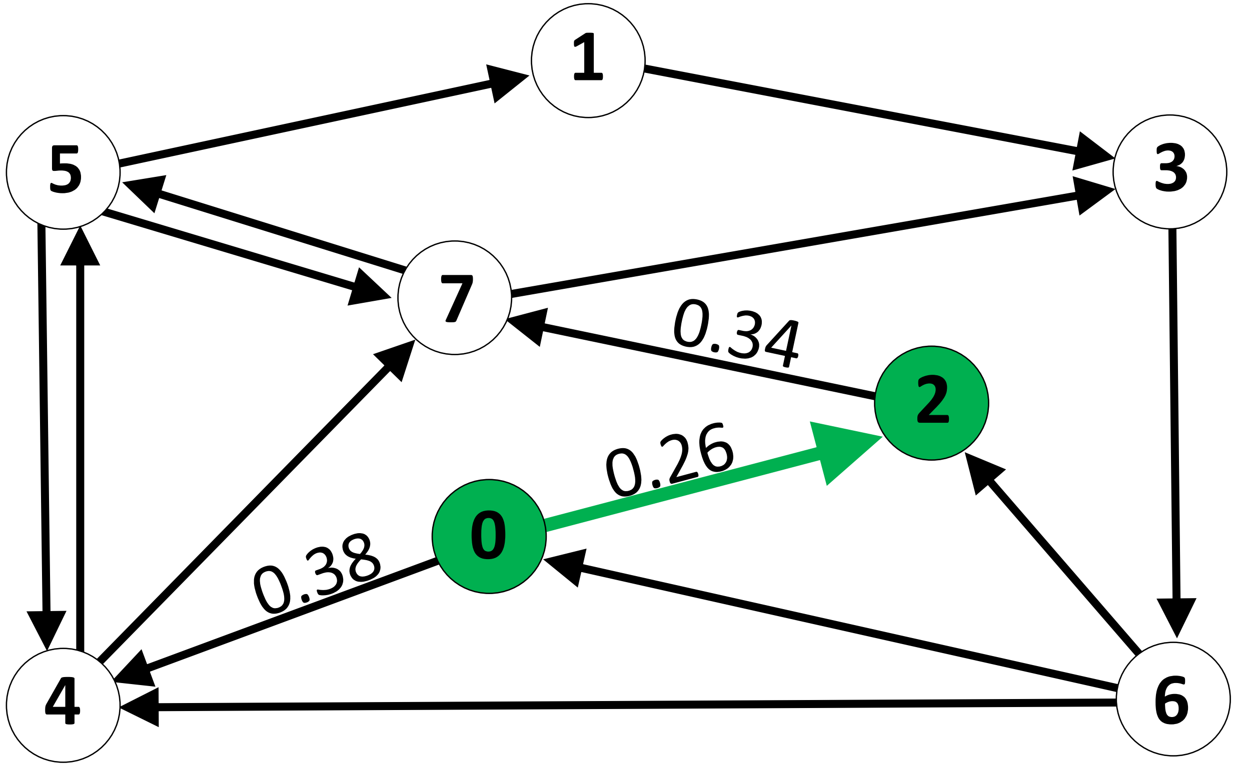
7 (0.60)

vertex (distance)

What can we say about the shortest path from 0 to 4?

Shortest Path

queue top = 4 (0.38)



	Distance from 0
0	0
1	∞
2	0.26
3	∞
4	0.38
5	∞
6	∞
7	0.60

	Previous vertex
0	-
1	
2	0
3	
4	0
5	
6	
7	2

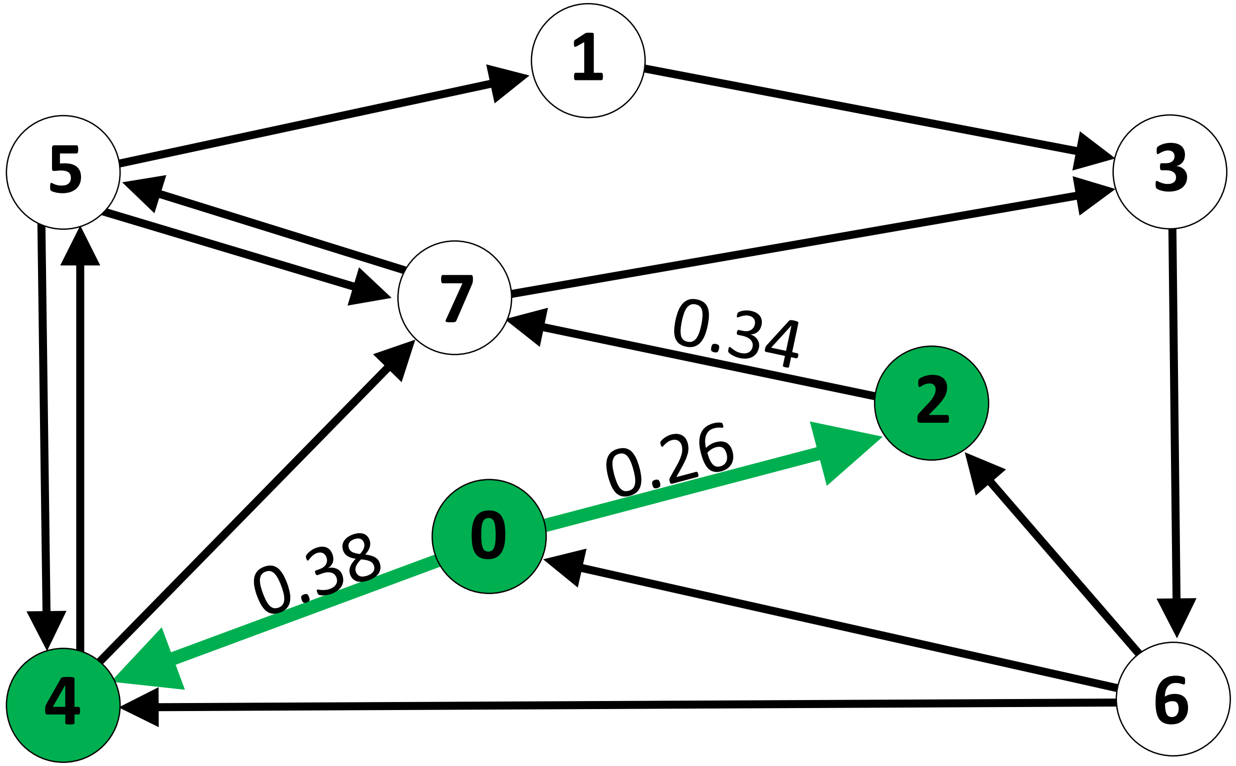
Priority queue
7 (0.60)

vertex (distance)

The 0 to 4 edge has to be the shortest path between 0 and 4, since any other path would go from 0 -> 2 -> 7 -> ? at cost at least 0.26 + 0.34 = 0.6 > 0.38

Shortest Path

queue top	= 4 (0.38)



	Distance from 0
0	0
1	∞
2	0.26
3	∞
4	0.38
5	∞
6	∞
7	0.60

	Previous vertex
0	-
1	
2	0
3	
4	0
5	
6	
7	2

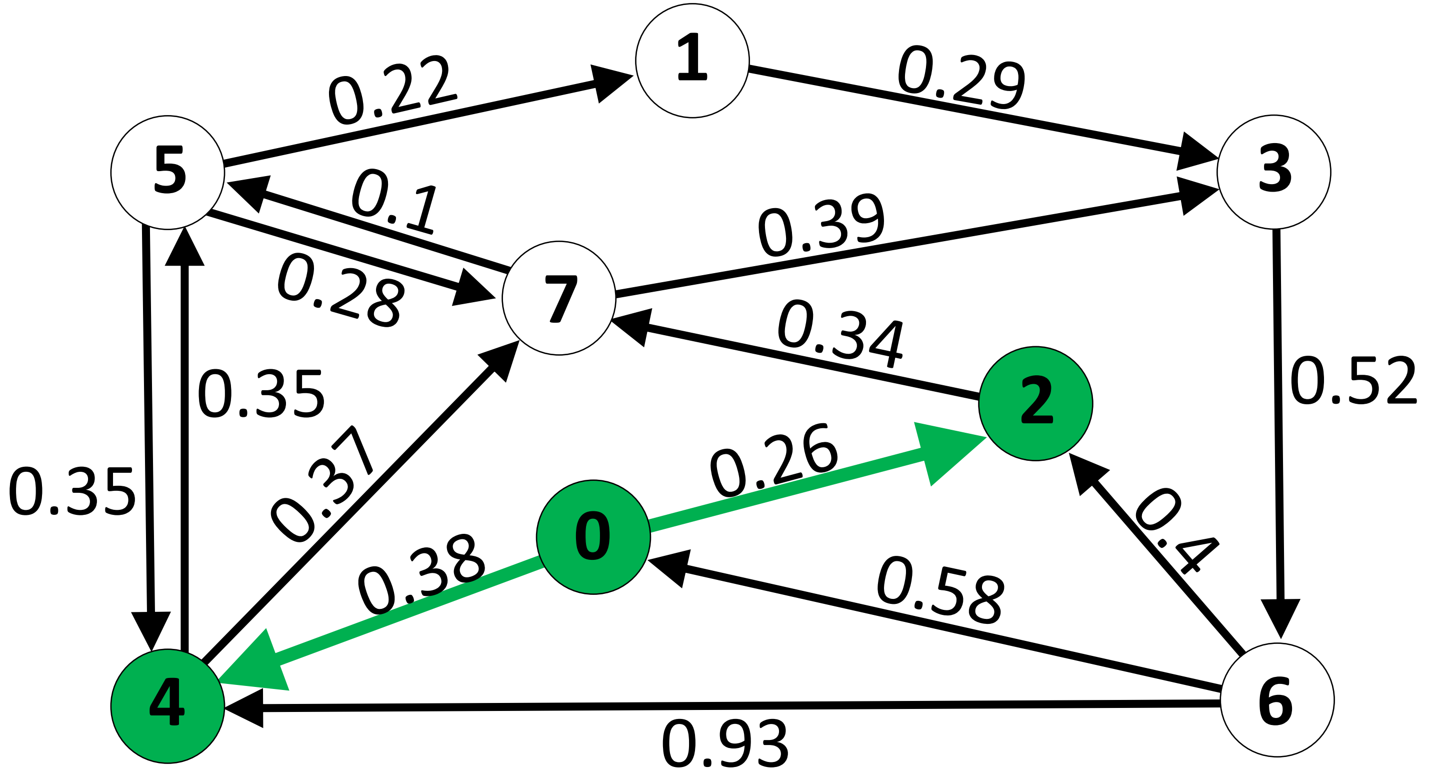
Priority queue
7 (0.60)

vertex (distance)

The 0 to 4 edge has to be the shortest path between 0 and 4, since any other path would go from 0 -> 2 -> 7 -> ? at cost at least 0.26 + 0.34 = 0.6 > 0.38

Shortest Path

queue top = 4 (0.38)



Add neighbors to queue/previous.

	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	0.38	0
5	∞	
6	∞	
7	0.60	2

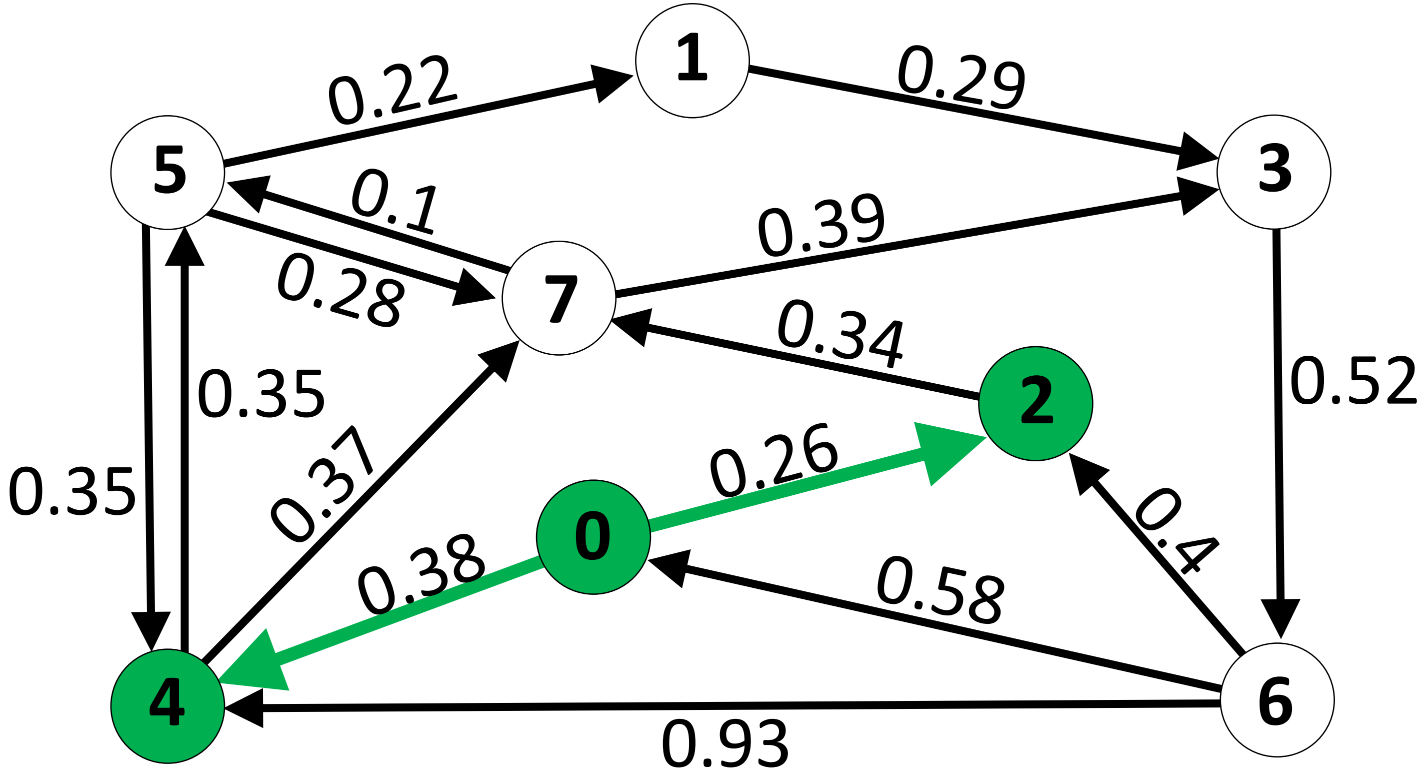
Priority queue

7 (0.60)

vertex (distance)

Shortest Path

queue top	= 4 (0.38)



Add neighbors to queue/previous.

	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	0.38	0
5	0.73	4
6	∞	
7	0.60	2

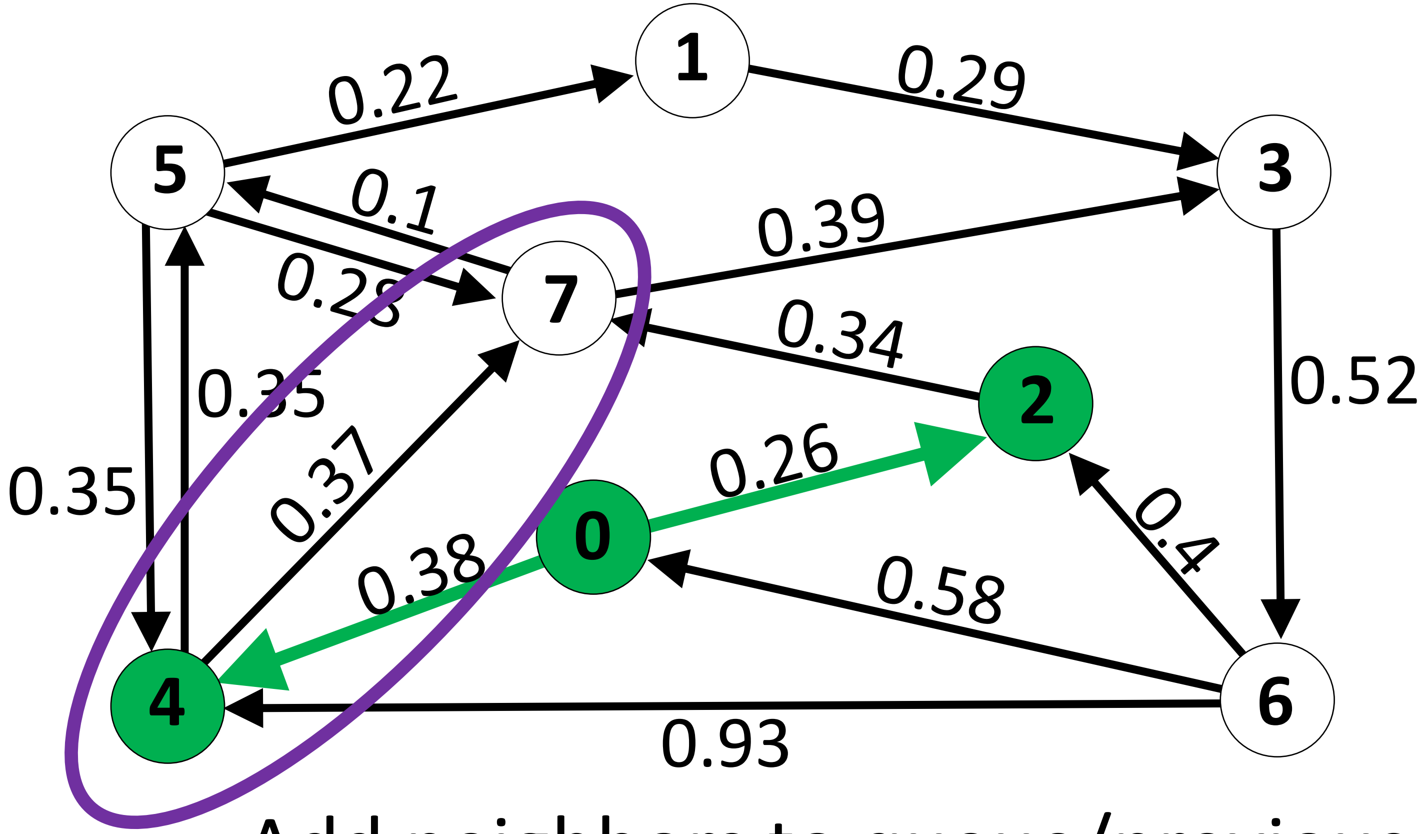
Priority queue

7 (0.60)
5 (0.73)

vertex (distance)

Shortest Path

queue top = 4 (0.38)



	Distance from 0
0	0
1	∞
2	0.26
3	∞
4	0.38
5	0.73
6	∞
7	0.60

	Previous vertex
0	-
1	
2	0
3	
4	0
5	4
6	
7	2

Priority queue

7 (0.60)
5 (0.73)

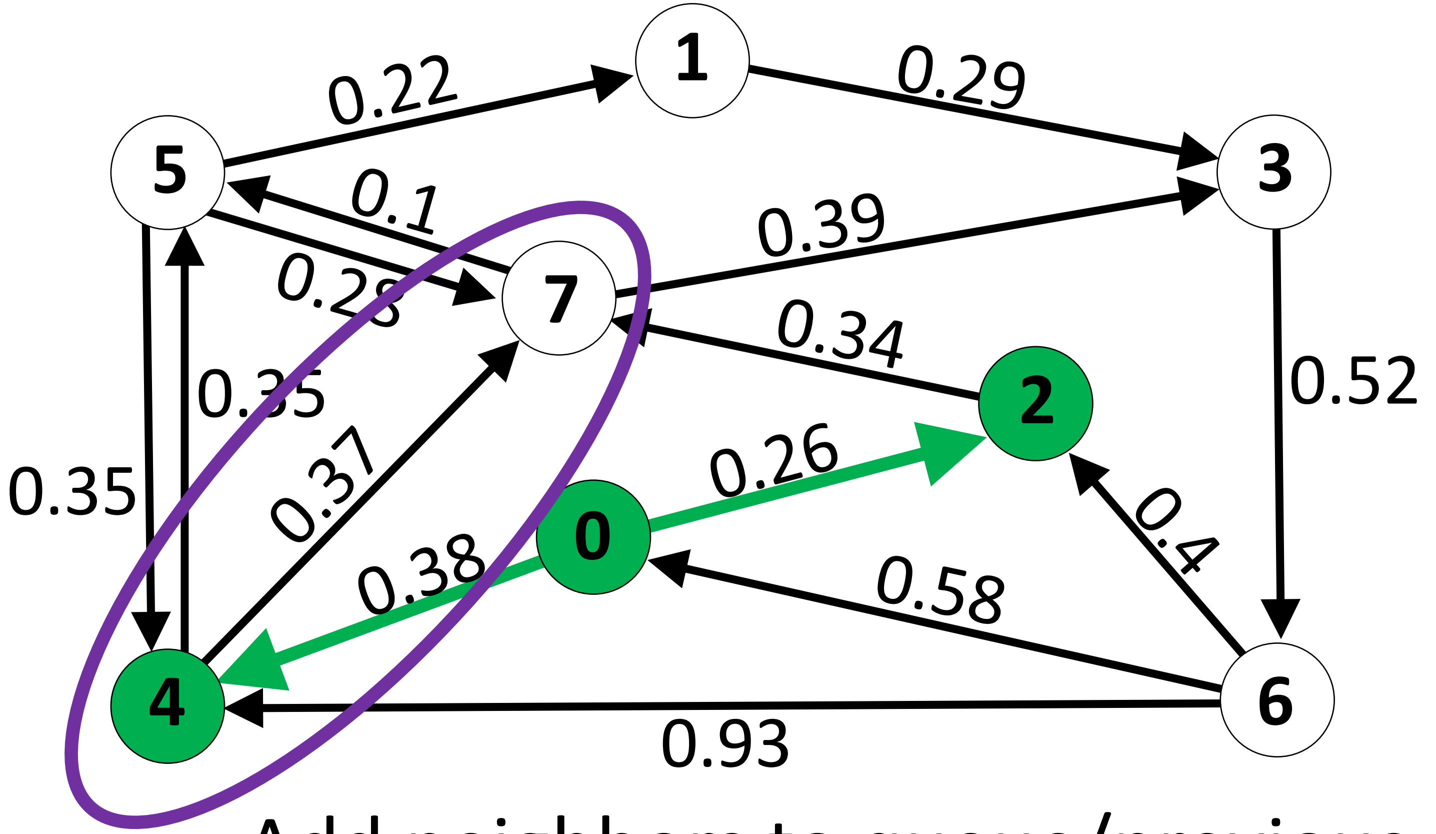
vertex (distance)

Add neighbors to queue/previous.

We have another route to 7!

Shortest Path

queue top = 4 (0.38)



Vertex	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	0.38	0
5	0.73	4
6	∞	
7	0.60	2

Priority queue

7 (0.60)
5 (0.73)

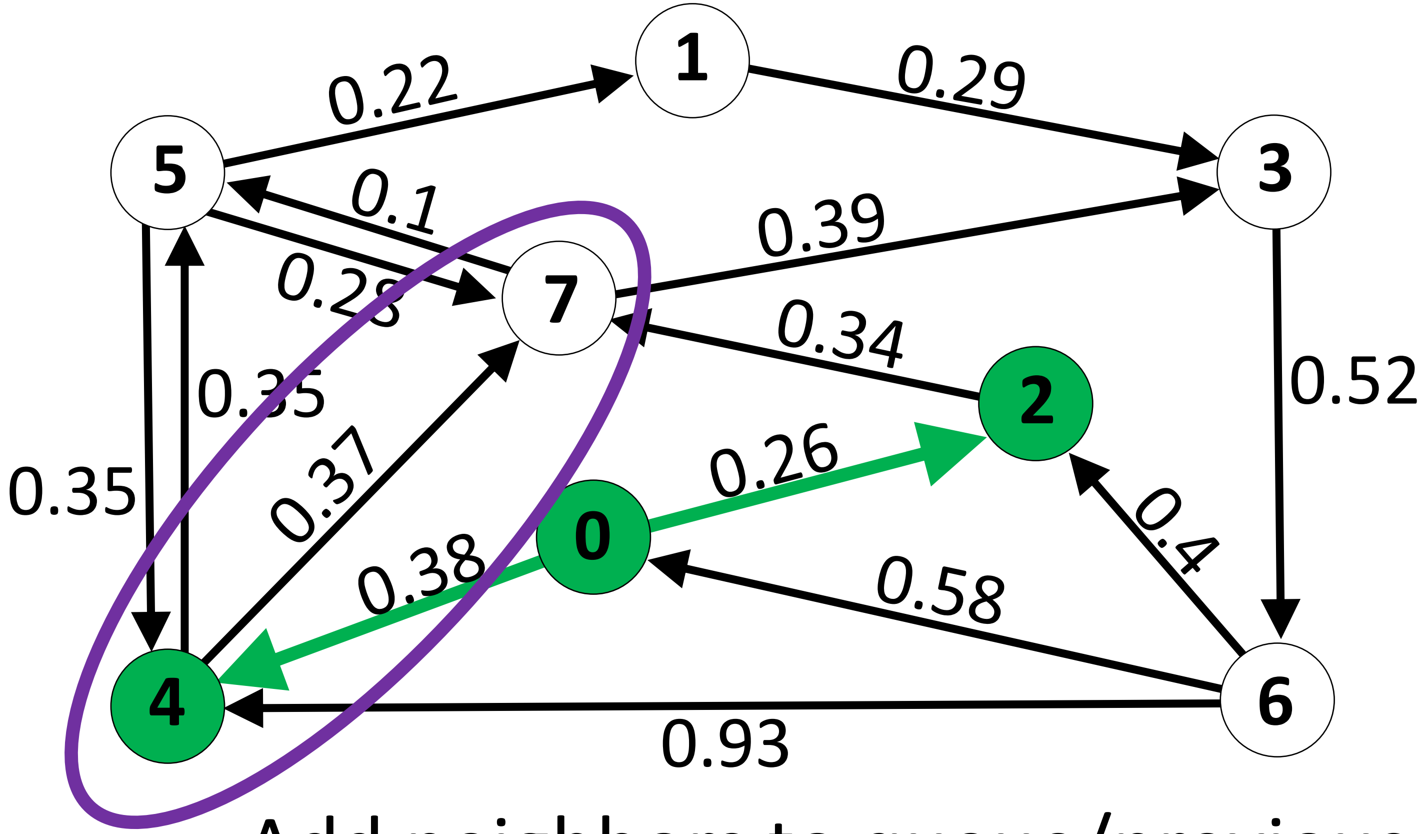
vertex (distance)

Add neighbors to queue/previous.

We have another route to 7! Check to see if it is shorter!

Shortest Path

queue top = 4 (0.38)



	Distance from 0
0	0
1	∞
2	0.26
3	∞
4	0.38
5	0.73
6	∞
7	0.60

	Previous vertex
0	-
1	
2	0
3	
4	0
5	4
6	
7	2

Priority queue

7 (0.60)
5 (0.73)

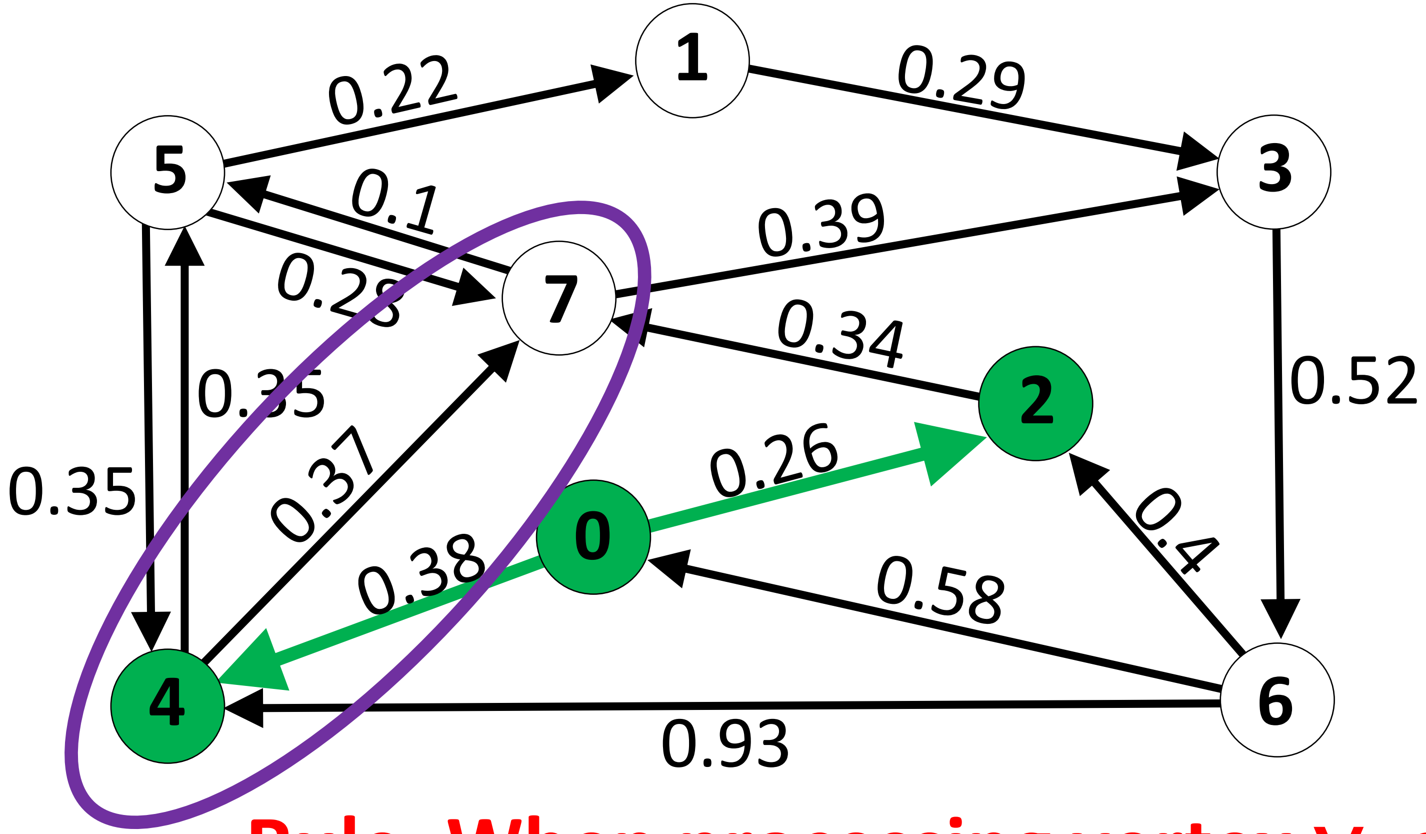
vertex (distance)

Add neighbors to queue/previous.

We have another route to 7! Check to see if it is shorter! It's not (0.38 + 0.37 = 0.75 > 0.60).

Shortest Path

queue top	= 4 (0.38)



Vertex	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	0.38	0
5	0.73	4
6	∞	
7	0.60	2

Priority queue

7 (0.60)
5 (0.73)

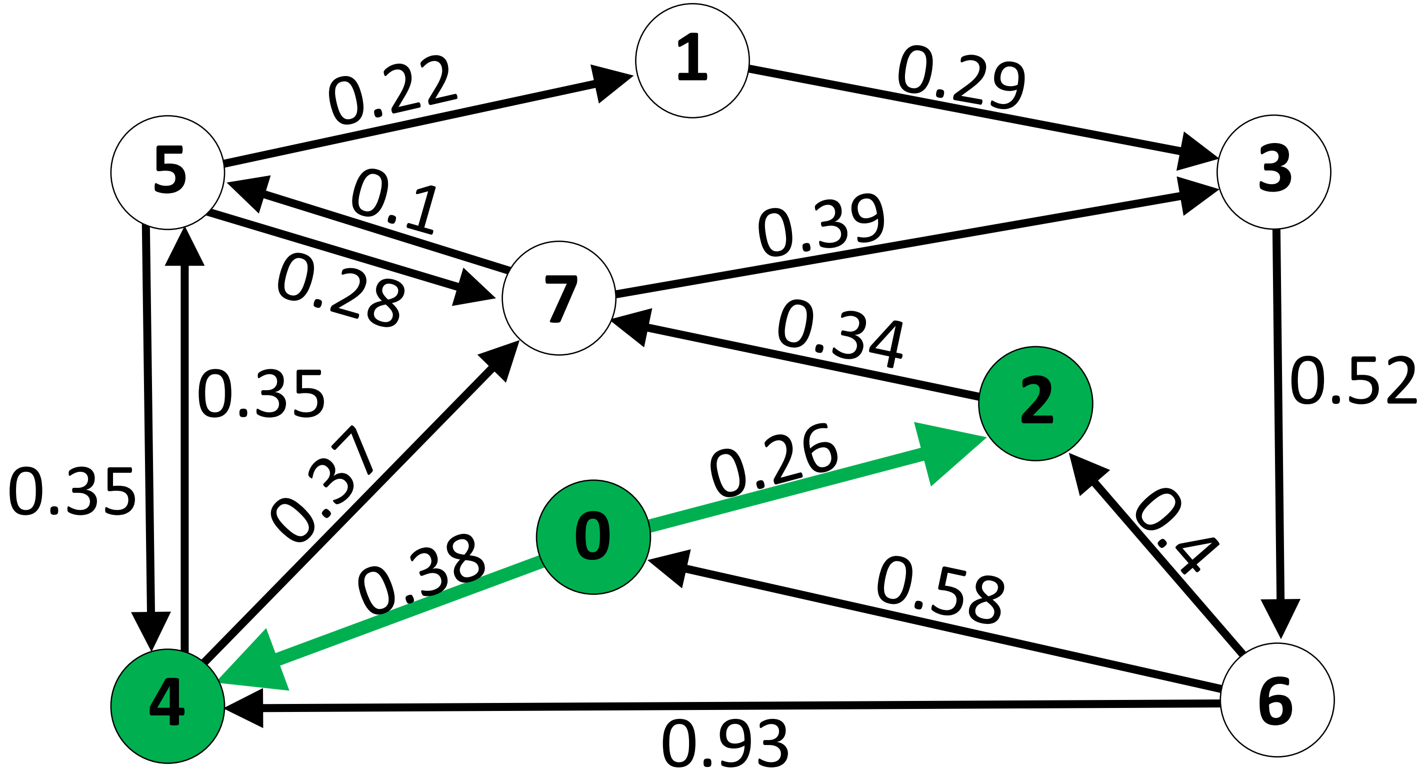
vertex (distance)

Rule: When processing vertex v, only add/modify queue for neighbor u if and only if:

```
distance[v] + weight(v, u) < distance[u]
```

Shortest Path

queue top =



Repeat.

	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	∞	
4	0.38	0
5	0.73	4
6	∞	
7	0.60	2

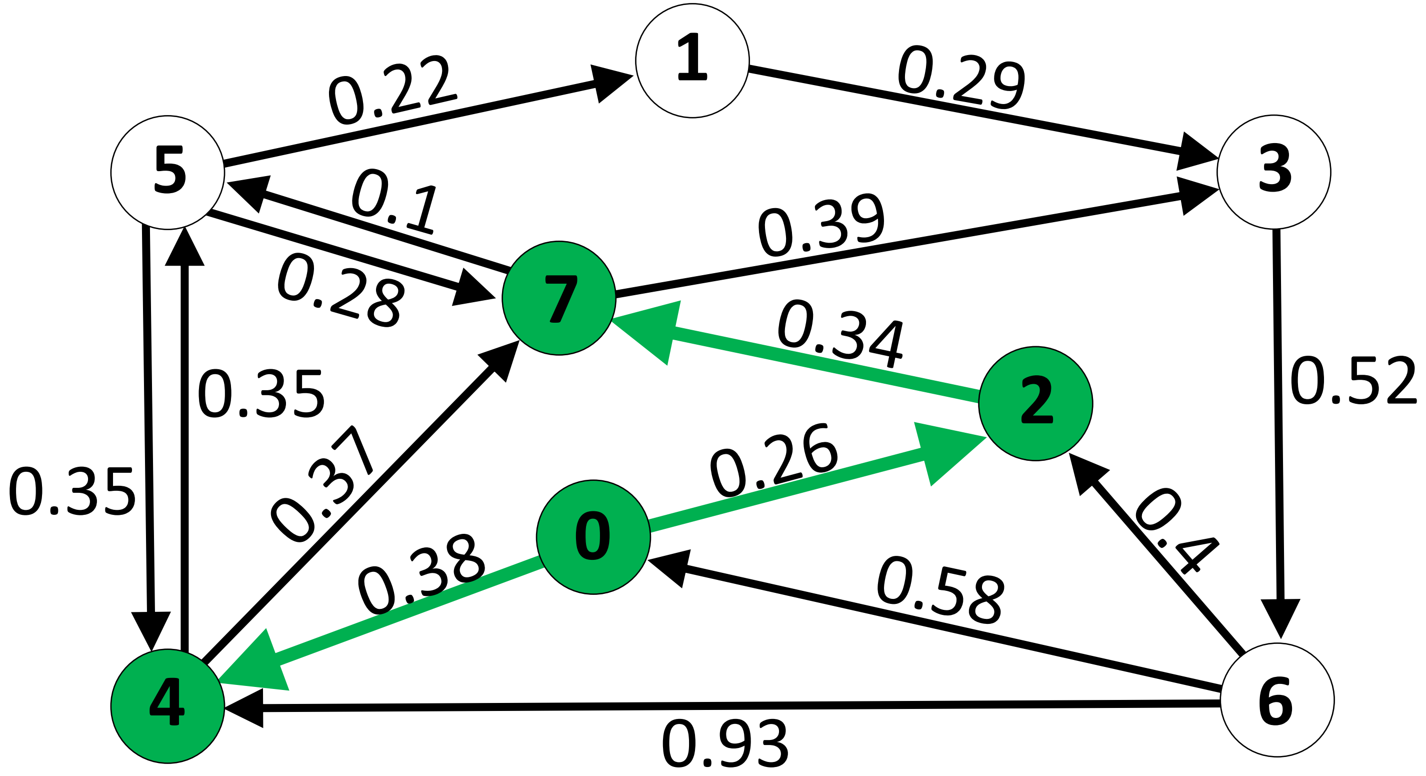
Priority queue

7 (0.60)
5 (0.73)

vertex (distance)

Shortest Path

queue top = 7 (0.60)



Repeat.

	Distance from 0
0	0
1	∞
2	0.26
3	0.99
4	0.38
5	0.73
6	∞
7	0.60

	Previous vertex
0	-
1	
2	0
3	7
4	0
5	4
6	
7	2

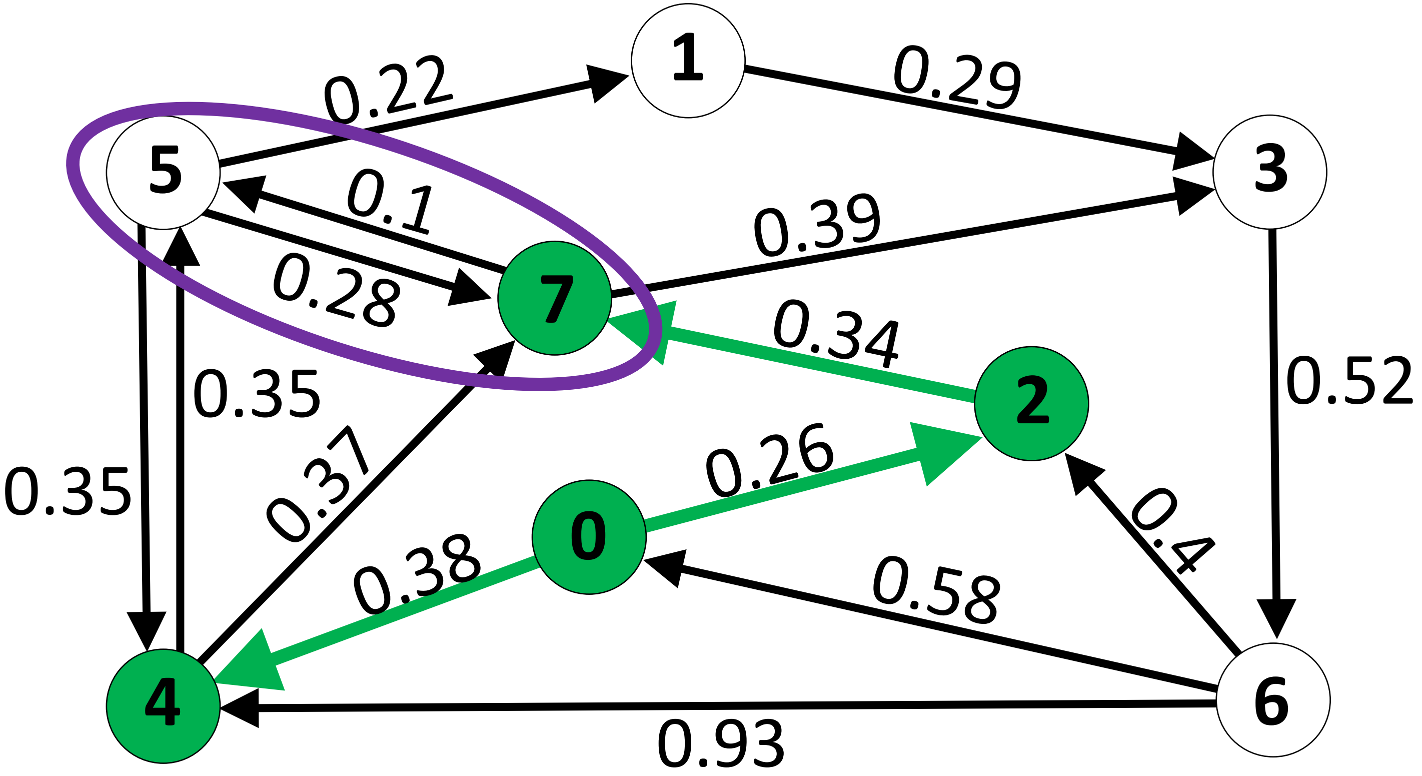
Priority queue

5 (0.73)
3 (0.99)

vertex (distance)

Shortest Path

queue top = 7 (0.60)



Distance from 0

0	0
1	∞
2	0.26
3	0.99
4	0.38
5	0.73
6	∞
7	0.60

Previous vertex

0	-
1	
2	0
3	7
4	0
5	4
6	
7	2

Priority queue

5 (0.73)
3 (0.99)

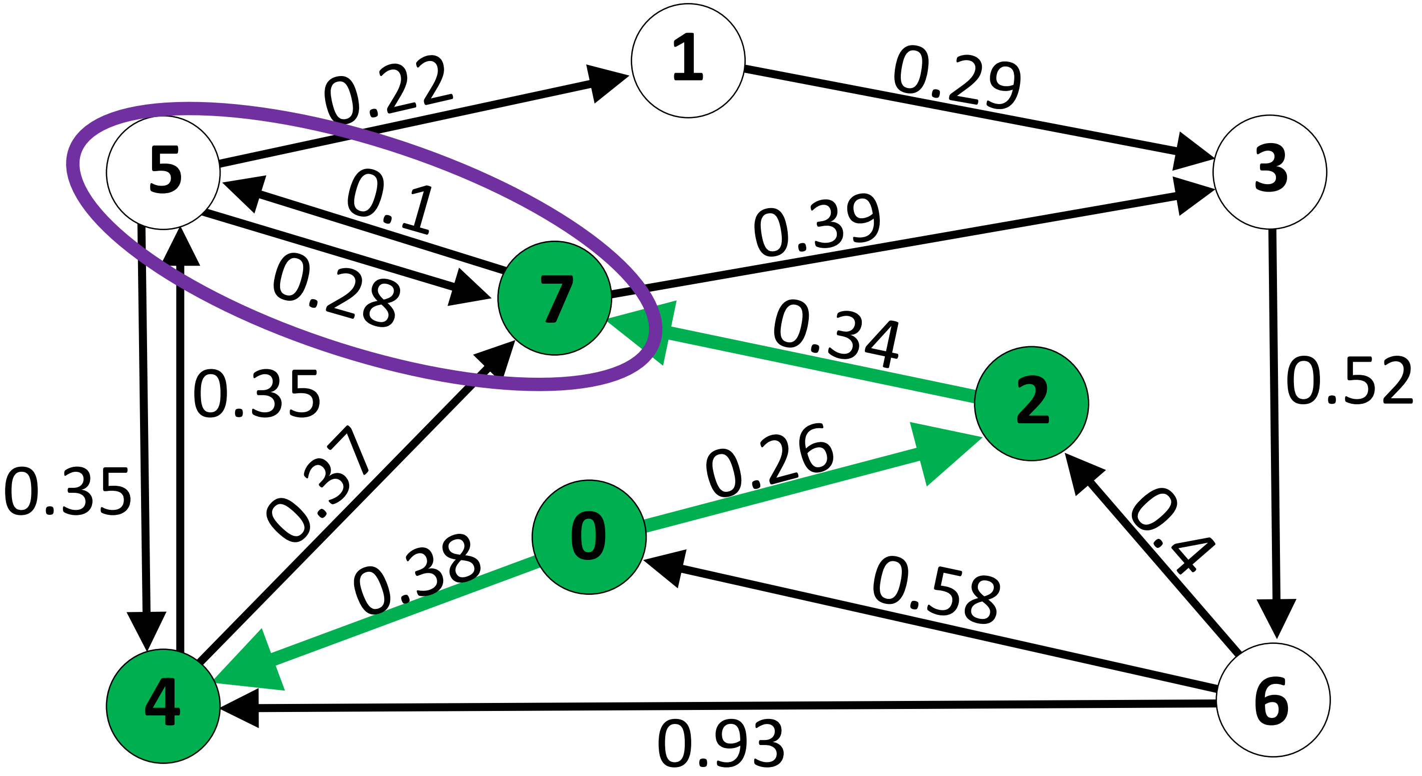
vertex (distance)

Repeat.

We have another route to 5, and at cost 0.7 < 0.73.

Shortest Path

queue top = 7 (0.60)



Vertex	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	0.99	7
4	0.38	0
5	0.73	4
6	∞	
7	0.60	2

Priority queue

5 (0.73)
3 (0.99)

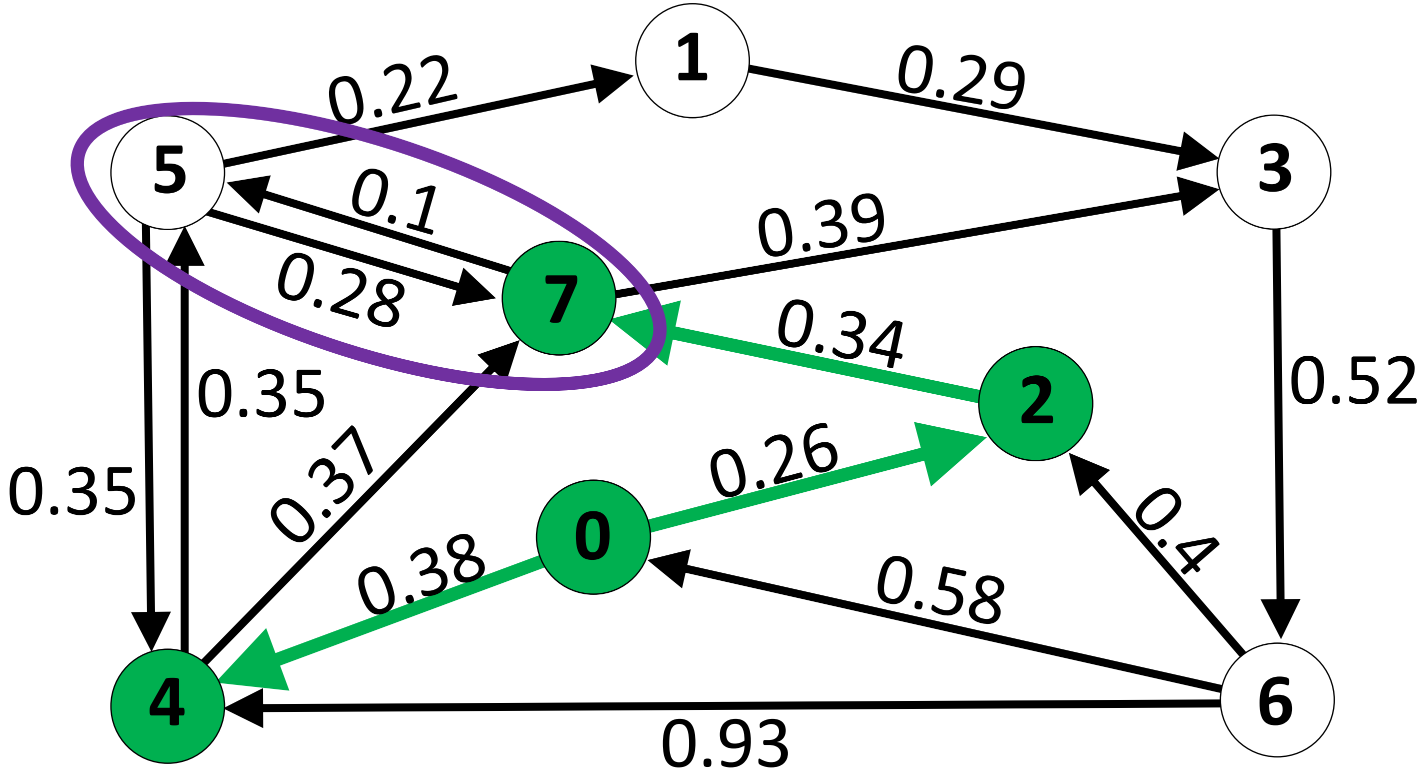
vertex (distance)

Repeat. **We have another route to 5, and at cost 0.7 < 0.73.**

i.e., `distance[v] + weight(v, u) < distance[u]`

Shortest Path

queue top = 7 (0.60)



	Distance from 0
0	0
1	∞
2	0.26
3	0.99
4	0.38
5	~~0.73~~ **0.70**
6	∞
7	0.60

	Previous vertex
0	-
1	
2	0
3	7
4	0
5	4 **7**
6	
7	2

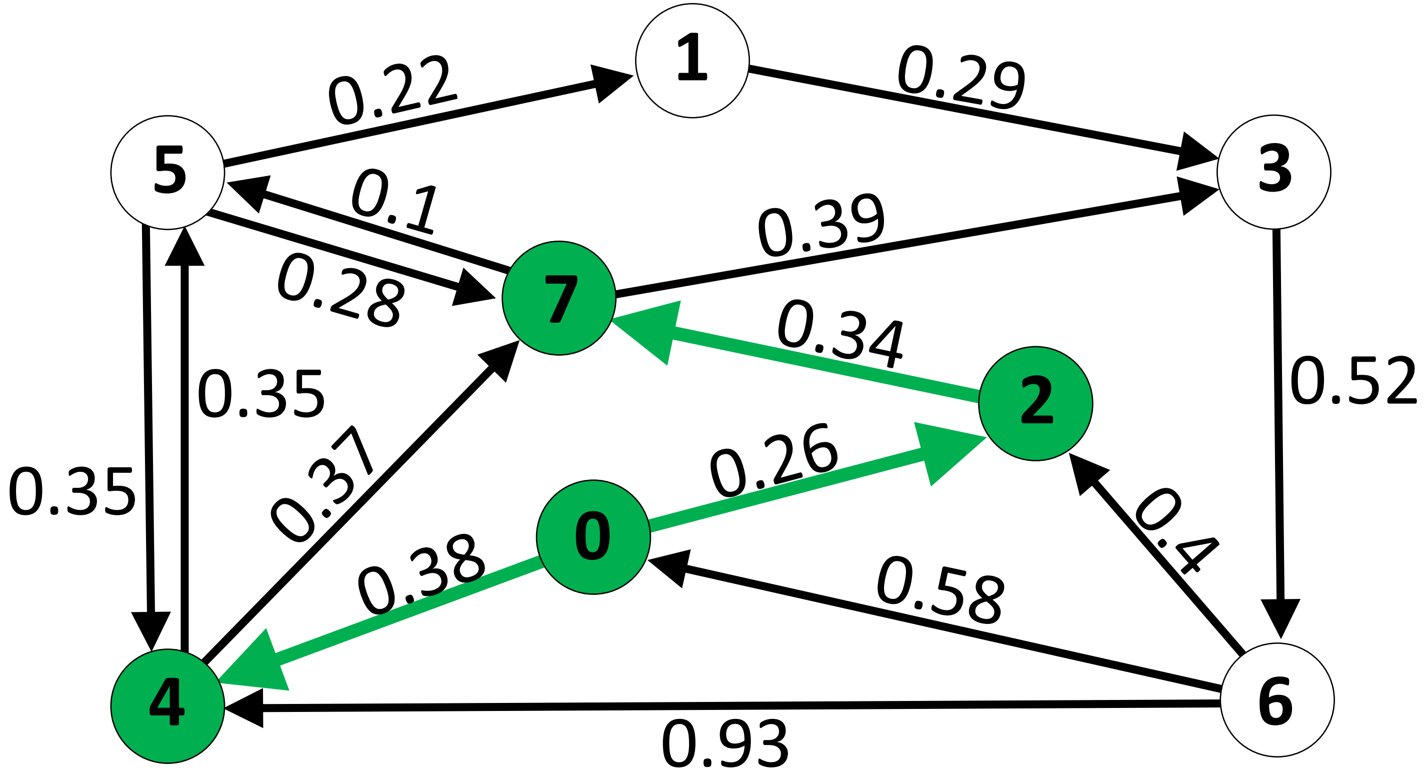
Priority queue
5 (~~0.73~~ **0.70**)
3 (0.99)

vertex (distance)

Repeat. **We have another route to 5, and at cost 0.7 < 0.73.**
So updated queue/previous/distance.

Shortest Path

queue top = 7 (0.60)



Distance from 0

0	0
1	∞
2	0.26
3	0.99
4	0.38
5	0.70
6	∞
7	0.60

Previous vertex

0	-
1	
2	0
3	7
4	0
5	7
6	
7	2

Priority queue

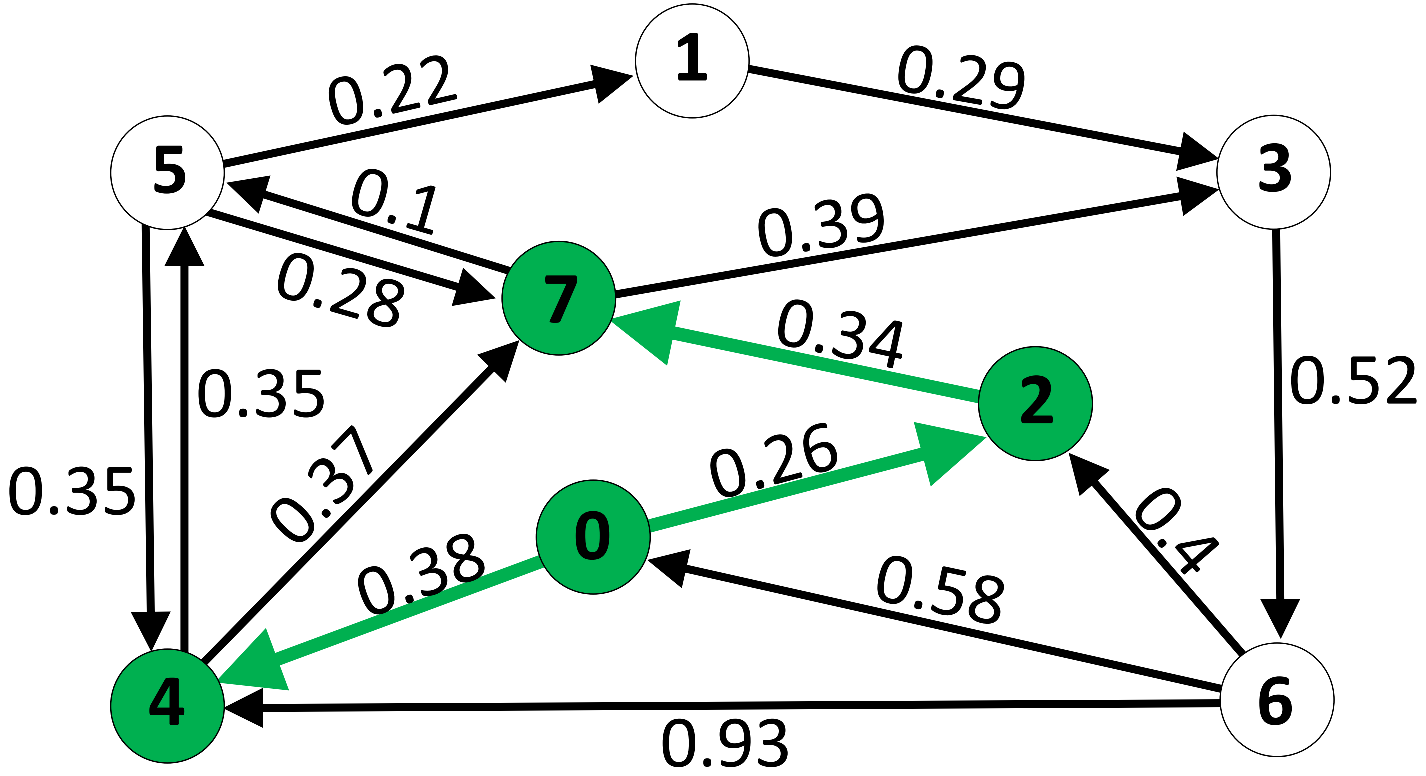
5 (0.70)
3 (0.99)

vertex (distance)

Repeat. **We have another route to 5, and at cost 0.7 < 0.73. So updated queue/previous/distance.**

Shortest Path

queue top	= 7 (0.60)



Repeat.

	Distance from 0	Previous vertex
0	0	-
1	∞	
2	0.26	0
3	0.99	7
4	0.38	0
5	0.70	7
6	∞	
7	0.60	2

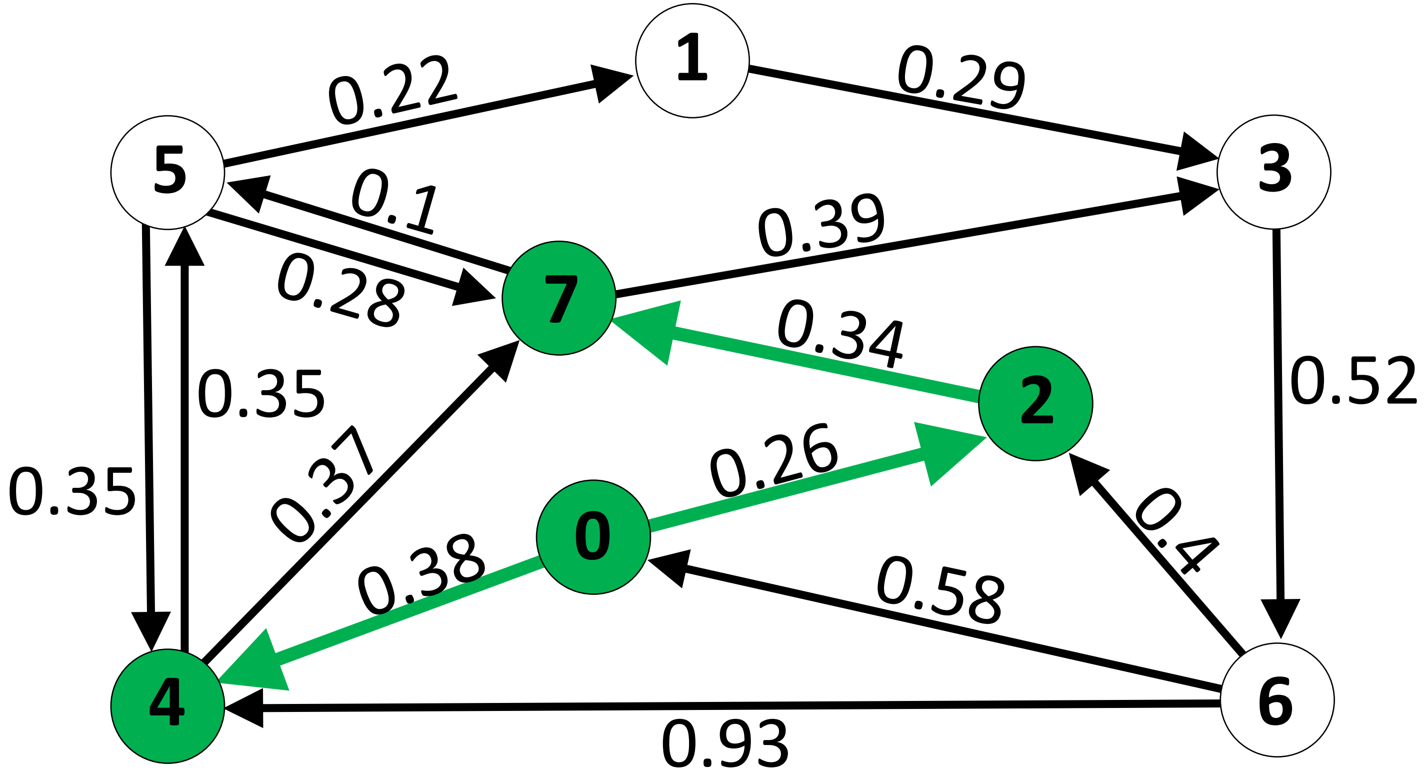
Priority queue

5 (0.70)
3 (0.99)

vertex (distance)

Shortest Path

queue top = 5 (0.70)



Repeat.

Distance from 0	
0	0
1	∞
2	0.26
3	0.99
4	0.38
5	0.70
6	∞
7	0.60

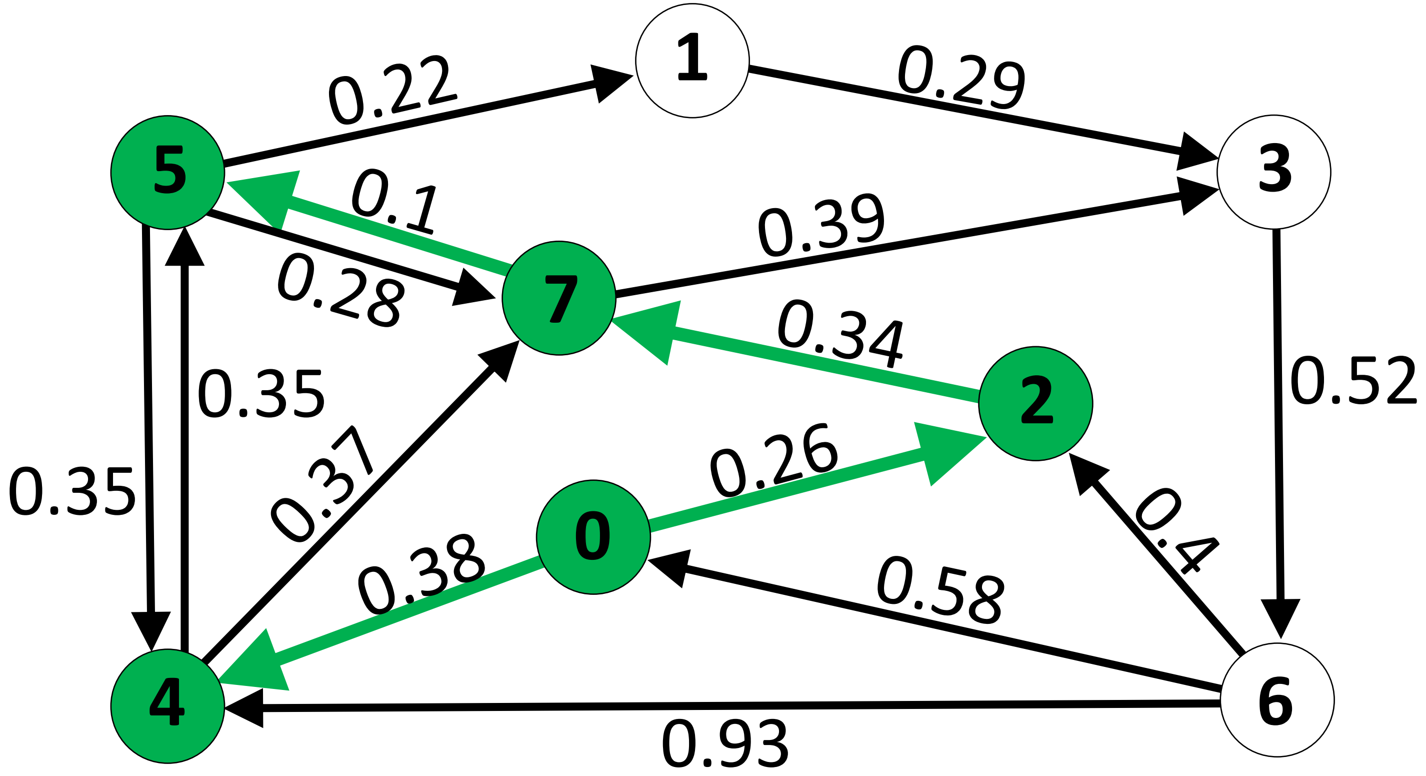
Previous vertex	
0	-
1	
2	0
3	7
4	0
5	7
6	
7	2

Priority queue
3 (0.99)

vertex (distance)

Shortest Path

queue top = 5 (0.70)



	Distance from 0
0	0
1	∞
2	0.26
3	0.99
4	0.38
5	0.70
6	∞
7	0.60

	Previous vertex
0	-
1	
2	0
3	7
4	0
5	7
6	
7	2

Priority queue

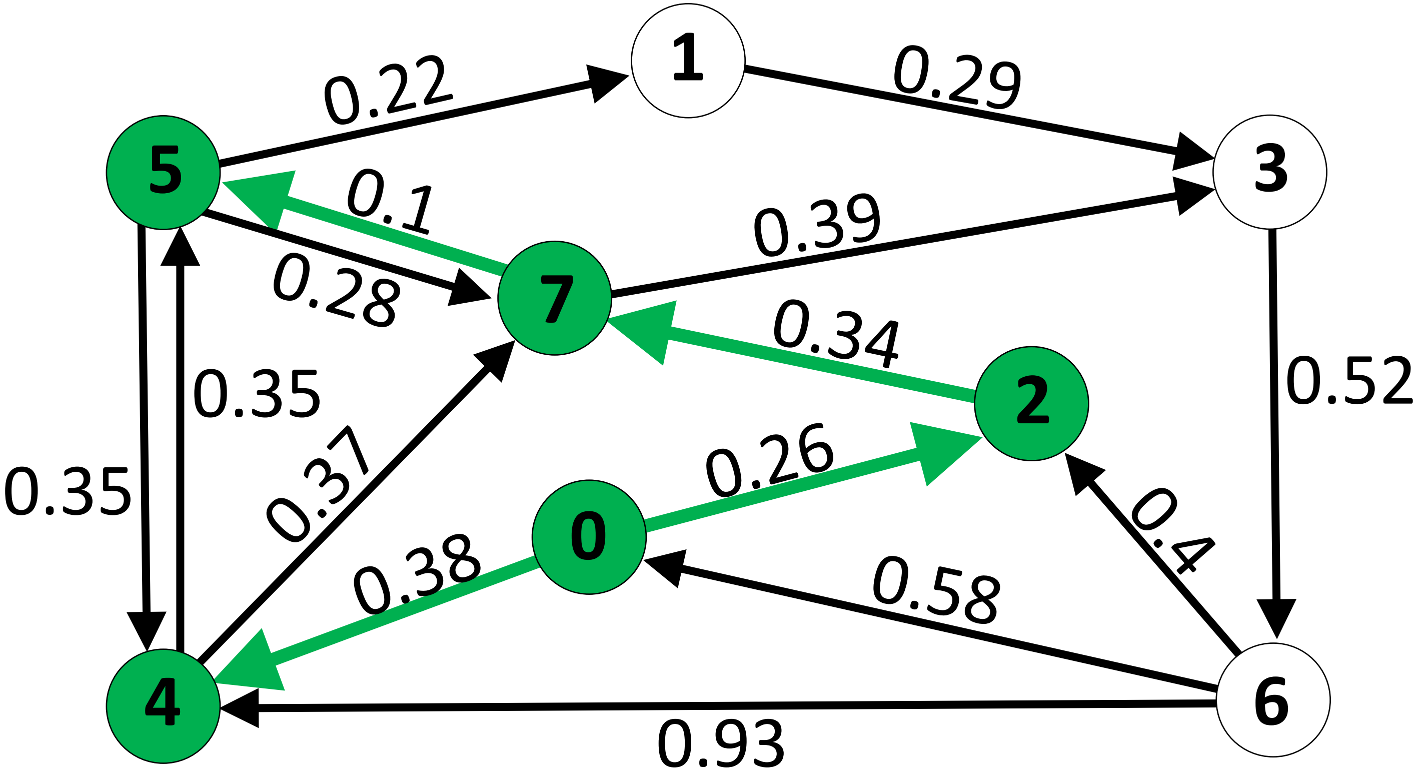
3 (0.99)

vertex (distance)

Repeat.

Shortest Path

queue top = 5 (0.70)



Repeat.

	Distance from 0
0	0
1	0.92
2	0.26
3	0.99
4	0.38
5	0.70
6	∞
7	0.60

	Previous vertex
0	-
1	5
2	0
3	7
4	0
5	7
6	
7	2

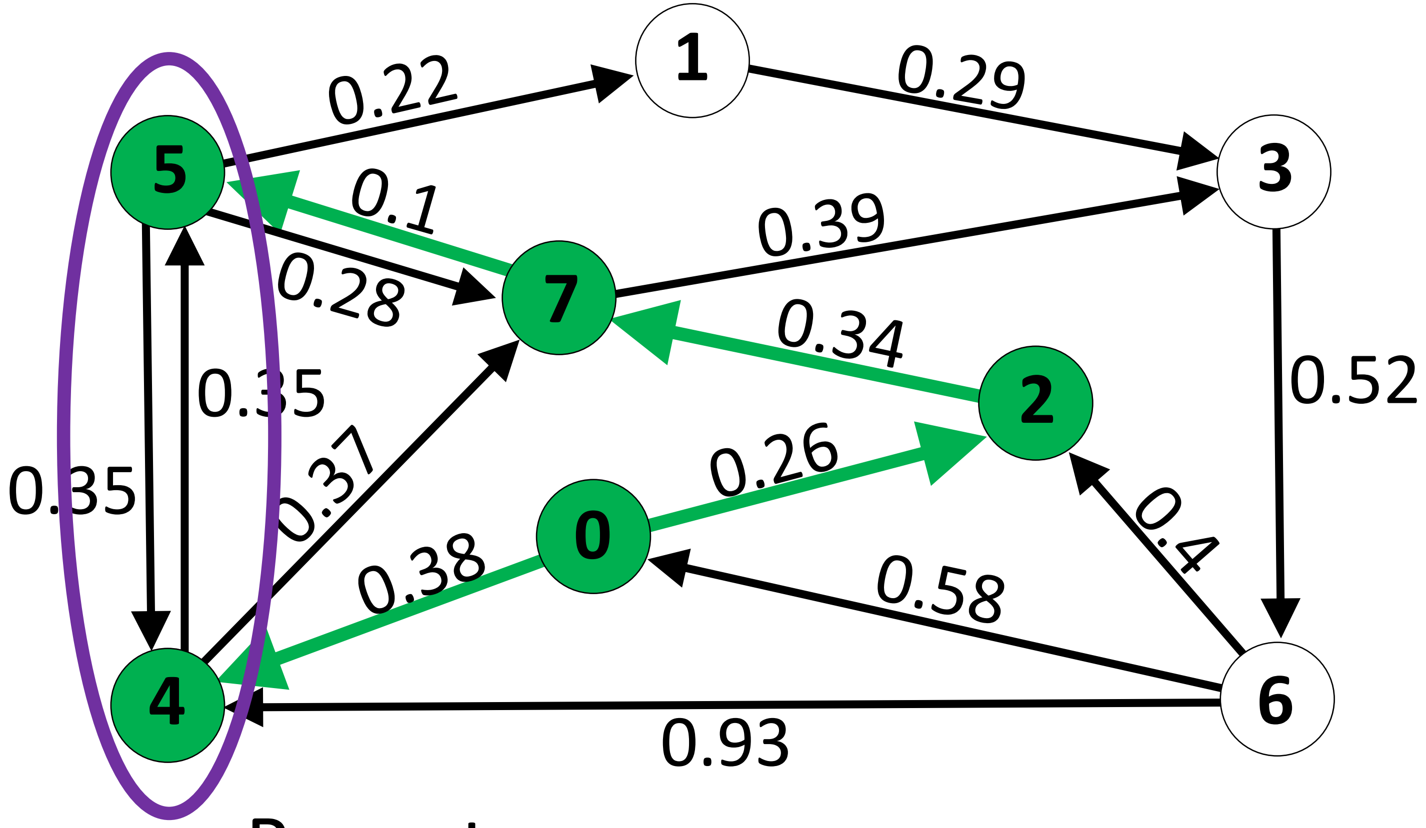
Priority queue

1 (0.92)
3 (0.99)

vertex (distance)

Shortest Path

queue top	= 5 (0.70)



	Distance from 0
0	0
1	0.92
2	0.26
3	0.99
4	0.38
5	0.70
6	∞
7	0.60

	Previous vertex
0	-
1	5
2	0
3	7
4	0
5	7
6	
7	2

Priority queue

1 (0.92)
3 (0.99)

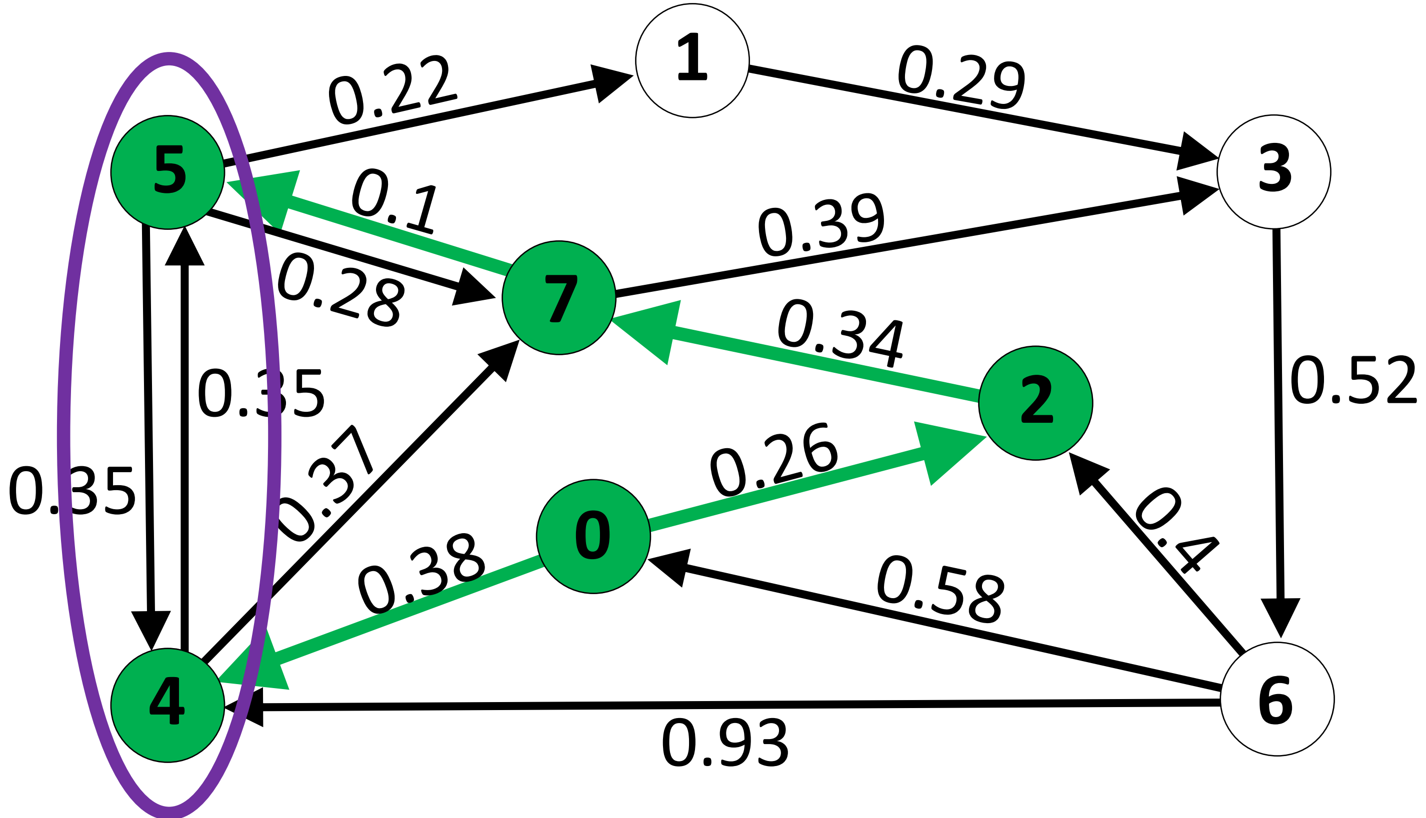
vertex (distance)

Repeat.

What about neighbor 4?

Shortest Path

queue top	= 5 (0.70)



Repeat.

What about neighbor 4? **distance[5] + weight(5, 4) = 0.70 + 0.35 = 1.05 ≮ 0.38 = distance[4]**

Distance from 0

0	0
1	0.92
2	0.26
3	0.99
4	0.38
5	0.70
6	∞
7	0.60

Previous vertex

0	-
1	5
2	0
3	7
4	0
5	7
6	
7	2

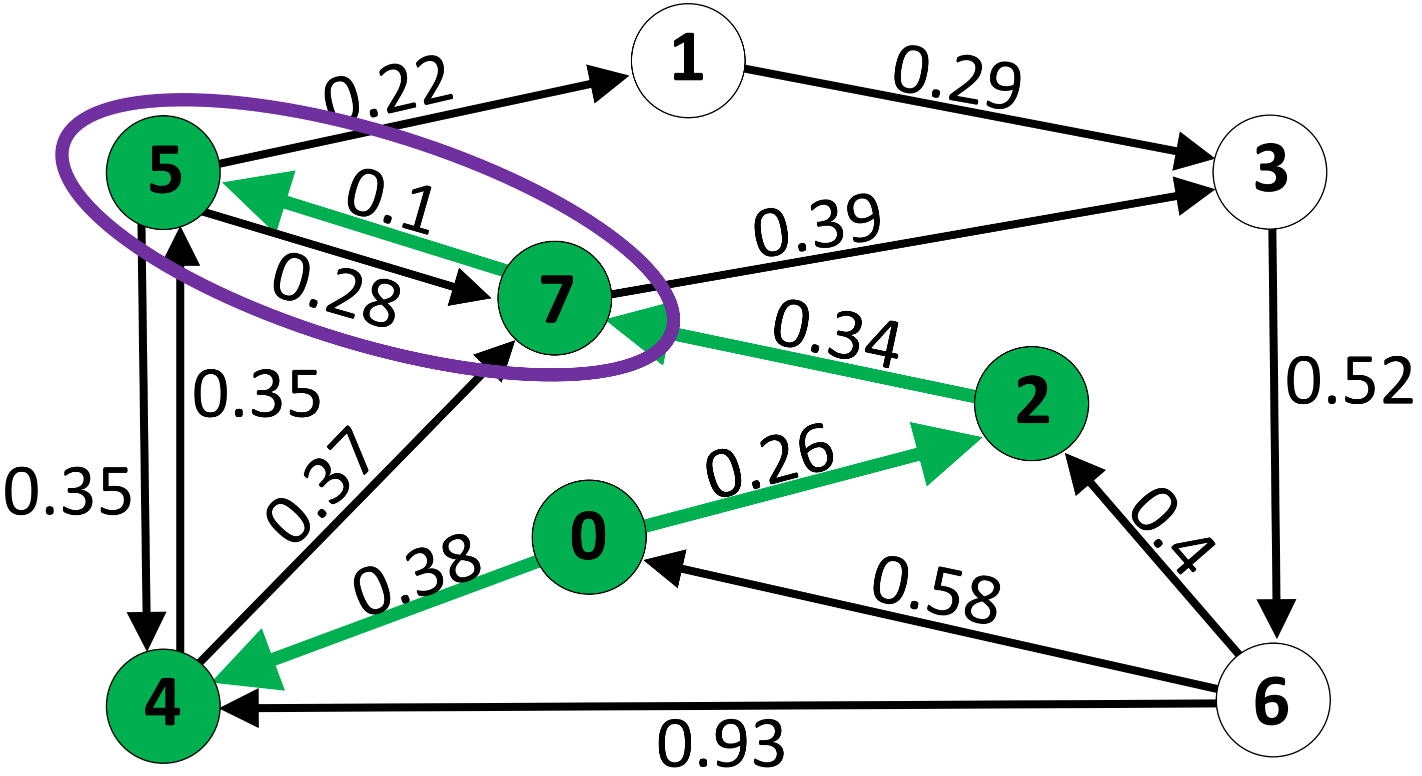
Priority queue

1 (0.92)
3 (0.99)

vertex (distance)

Shortest Path

queue top = 5 (0.70)



Vertex	Distance from 0	Previous vertex
0	0	-
1	0.92	5
2	0.26	0
3	0.99	7
4	0.38	0
5	0.70	7
6	∞	
7	0.60	2

Priority queue

1 (0.92)
3 (0.99)

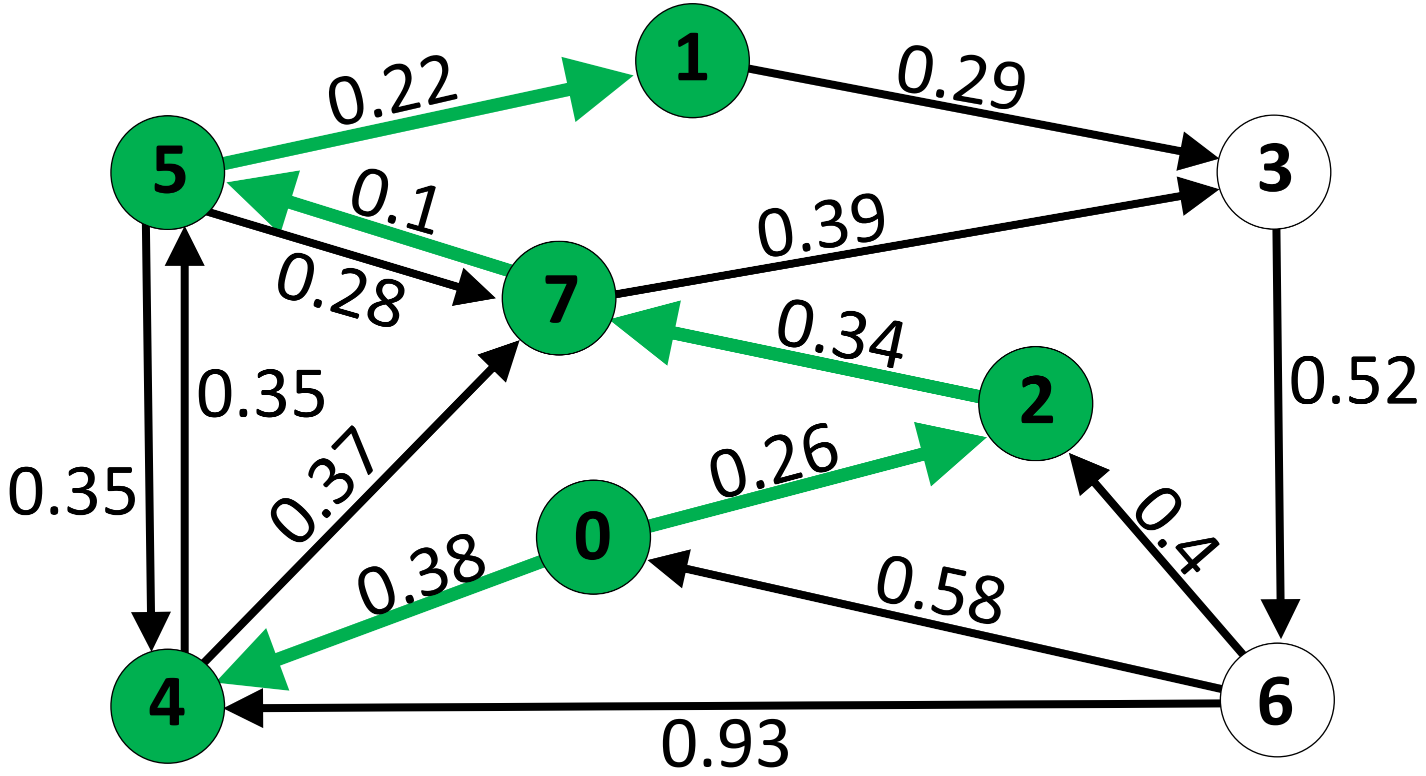
vertex (distance)

Repeat. **What about neighbor 7?**

distance[5] + weight(5, 7) = 0.70 + 0.28 = 0.98 $\not<$ 0.60 = distance[7]

Shortest Path

queue top = 1 (0.92)



Repeat.

	Distance from 0	Previous vertex
0	0	-
1	0.92	5
2	0.26	0
3	0.99	7
4	0.38	0
5	0.70	7
6	∞	
7	0.60	2

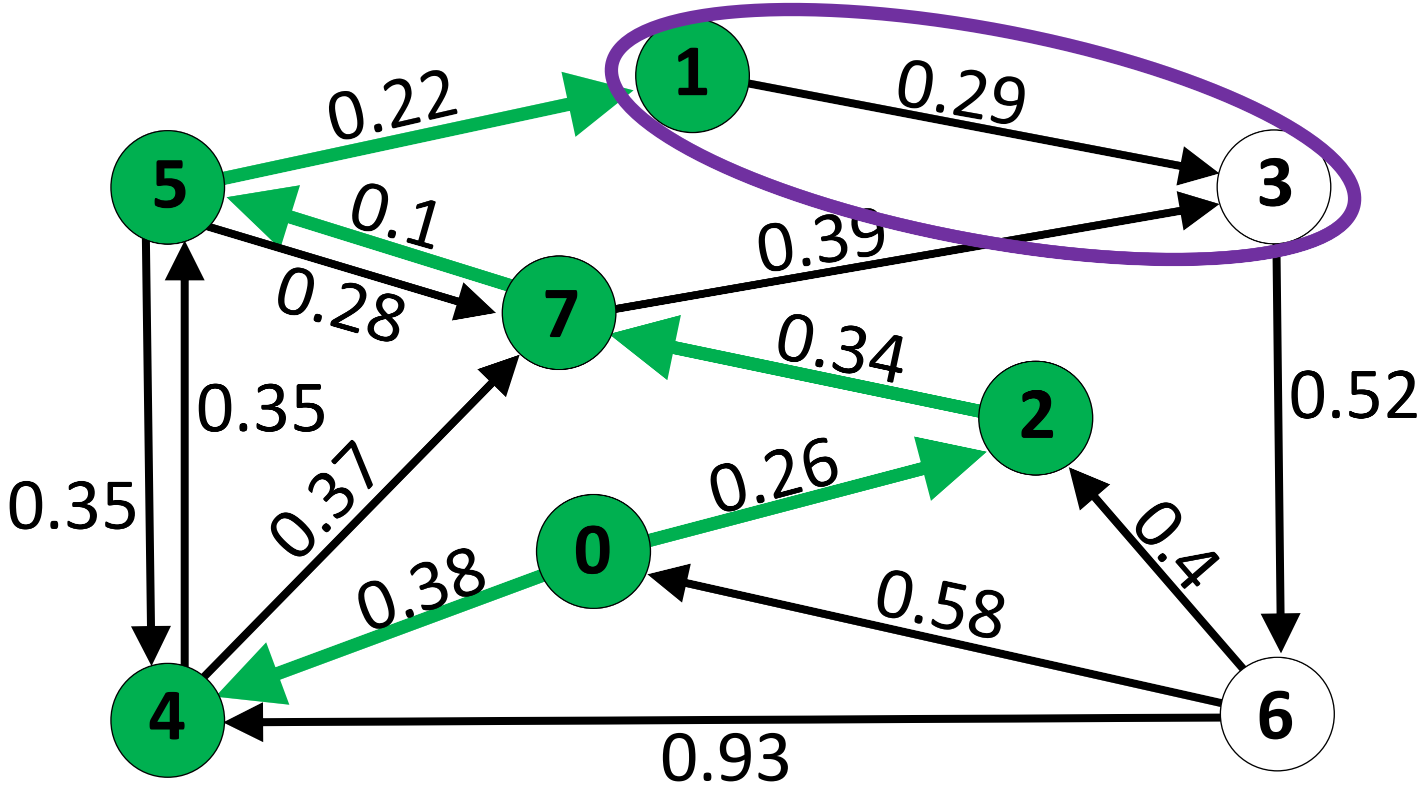
Priority queue

3 (0.99)

vertex (distance)

Shortest Path

queue top = 1 (0.92)



	Distance from 0	Previous vertex
0	0	-
1	0.92	5
2	0.26	0
3	0.99	7
4	0.38	0
5	0.70	7
6	∞	
7	0.60	2

Priority queue

3 (0.99)

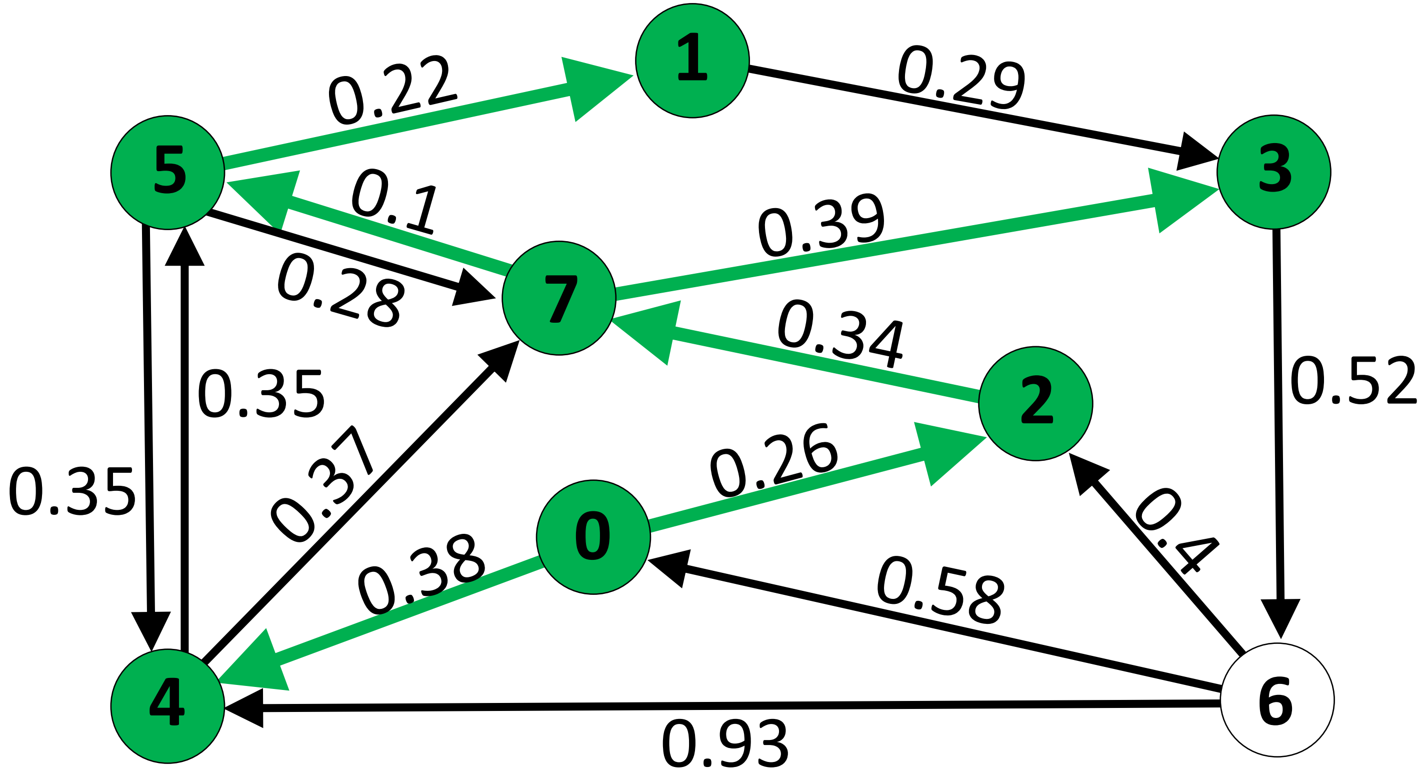
vertex (distance)

Repeat.

What about neighbor 3? 0.92 + 0.29 = 1.21 > 0.99

Shortest Path

queue top = 3 (0.99)



Repeat.

	Distance from 0
0	0
1	0.92
2	0.26
3	0.99
4	0.38
5	0.70
6	∞
7	0.60

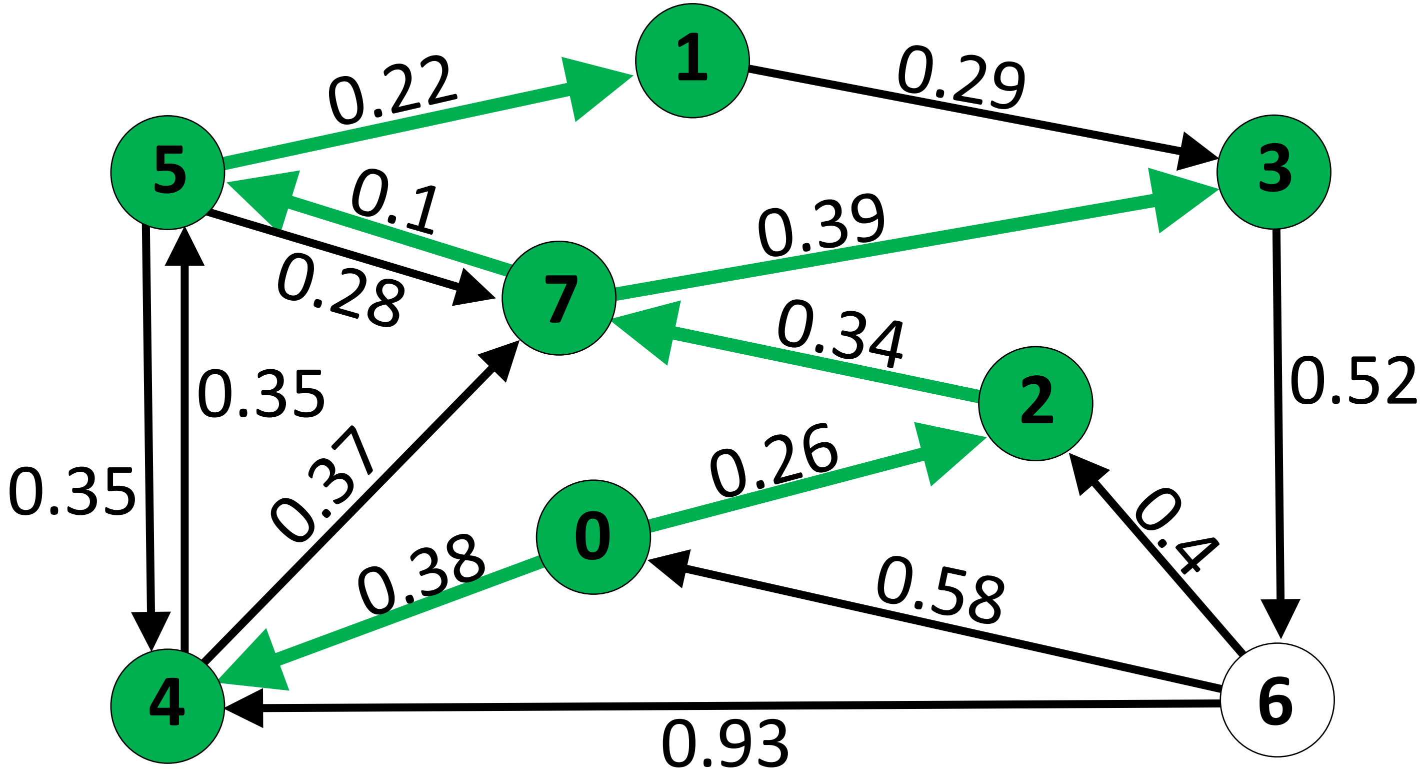
	Previous vertex
0	-
1	5
2	0
3	7
4	0
5	7
6	
7	2

Priority queue

vertex (distance)

Shortest Path

queue top = 3 (0.99)



Repeat.

	Distance from 0
0	0
1	0.92
2	0.26
3	0.99
4	0.38
5	0.70
6	1.51
7	0.60

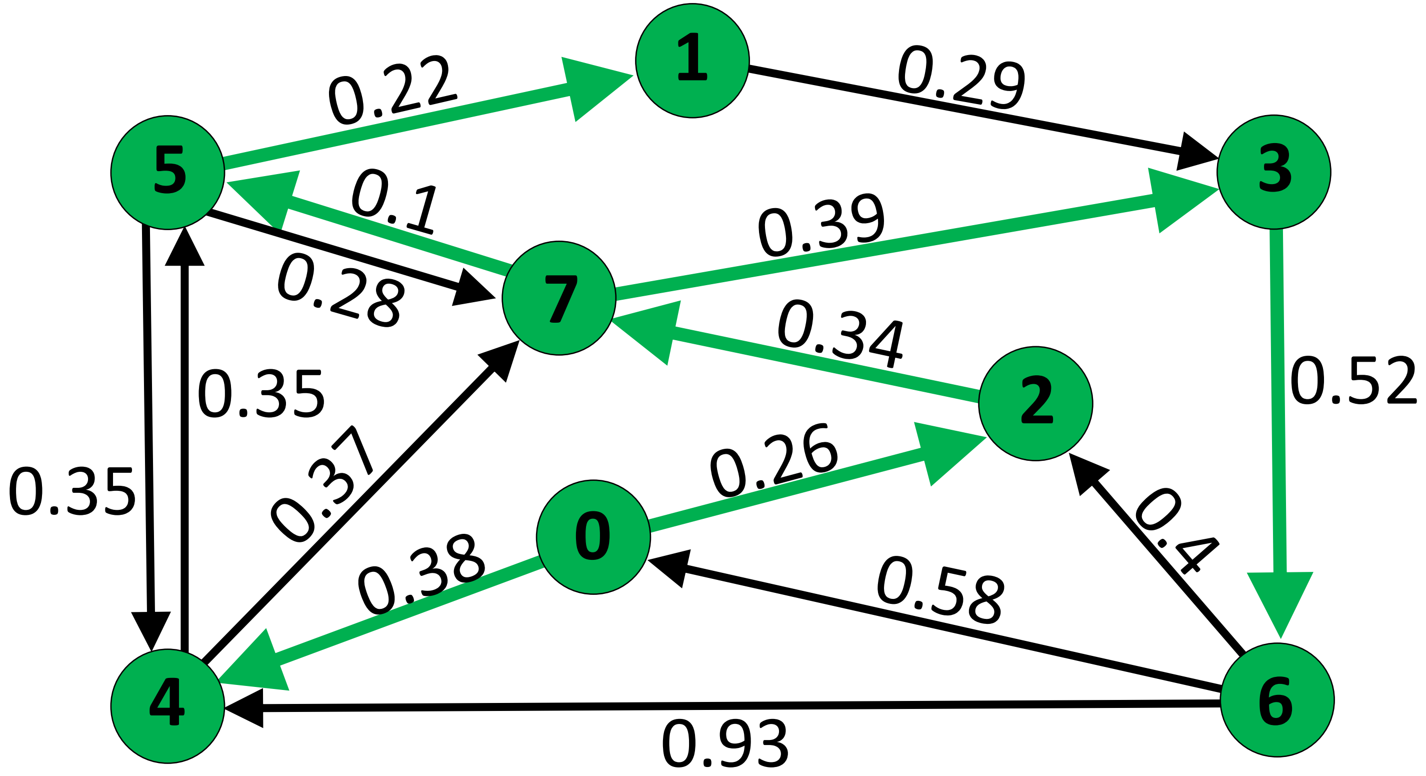
	Previous vertex
0	-
1	5
2	0
3	7
4	0
5	7
6	3
7	2

Priority queue
6 (1.51)

vertex (distance)

Shortest Path

queue top = 6 (1.51)



Repeat.

Distance from 0	
0	0
1	0.92
2	0.26
3	0.99
4	0.38
5	0.70
6	1.51
7	0.60

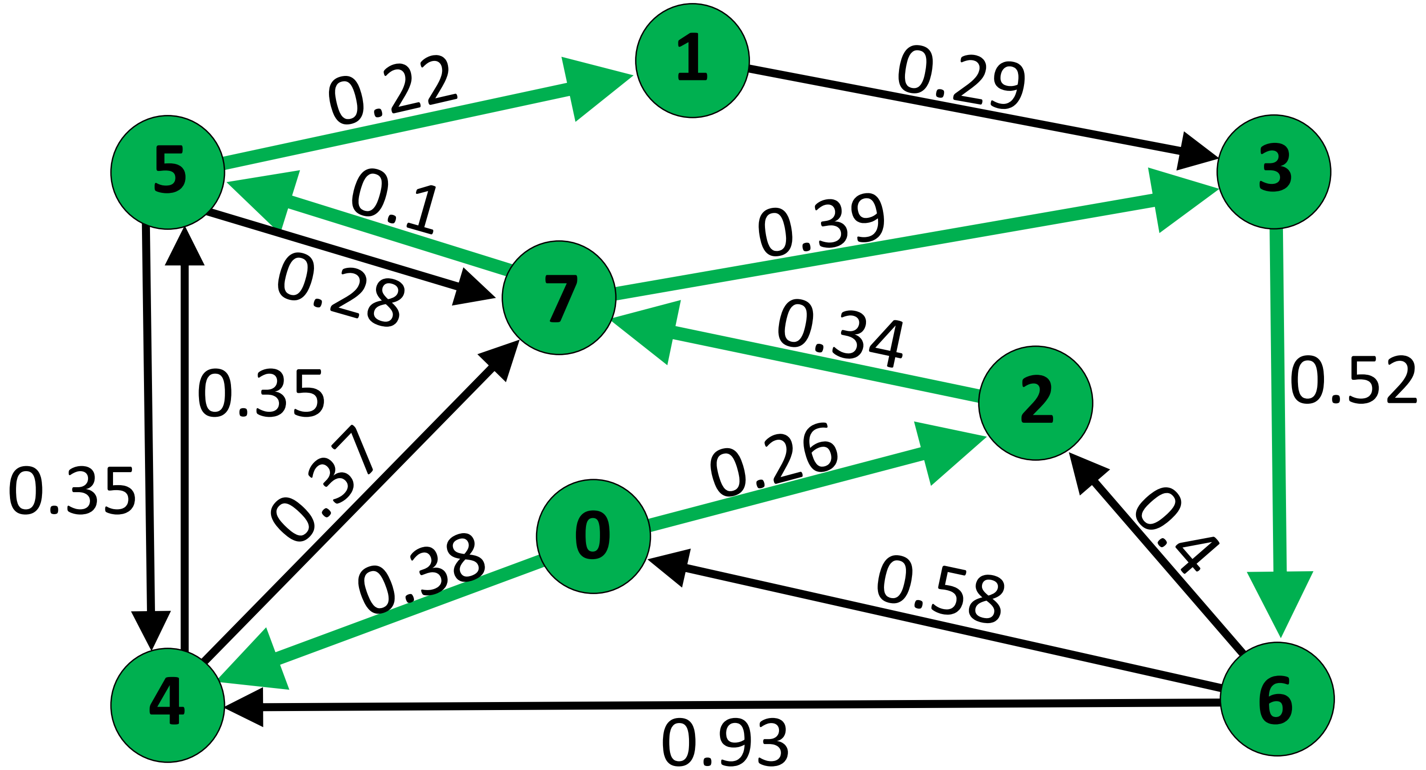
Previous vertex	
0	-
1	5
2	0
3	7
4	0
5	7
6	3
7	2

Priority queue

vertex (distance)

Shortest Path

queue top = 6 (1.51)



Repeat?

	Distance from 0
0	0
1	0.92
2	0.26
3	0.99
4	0.38
5	0.70
6	1.51
7	0.60

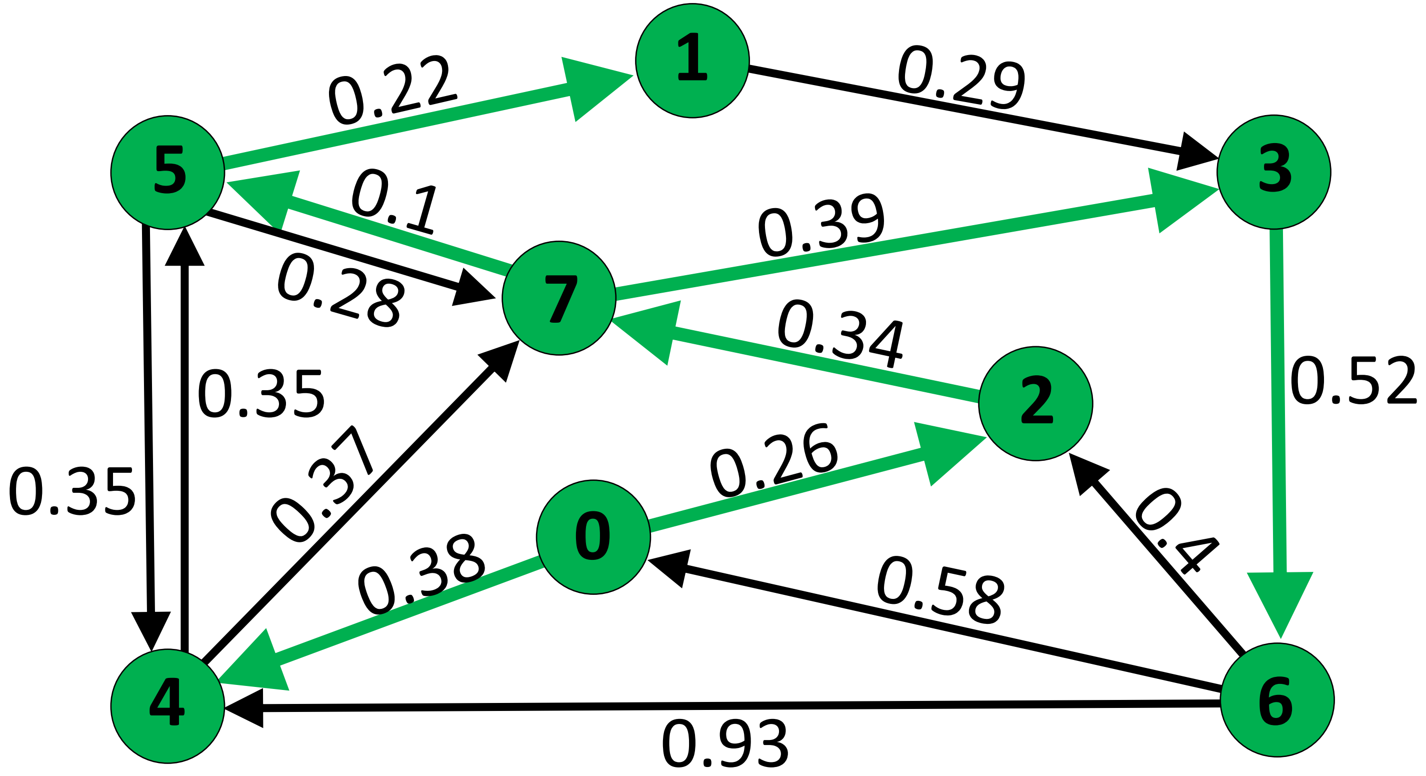
	Previous vertex
0	-
1	5
2	0
3	7
4	0
5	7
6	3
7	2

Priority queue

vertex (distance)

Shortest Path

queue top = 6 (1.51)



Repeat?

Neighbor 4? 1.51 + 0.93 > 0.38

Distance from 0

0	0
1	0.92
2	0.26
3	0.99
4	0.38
5	0.70
6	1.51
7	0.60

Previous vertex

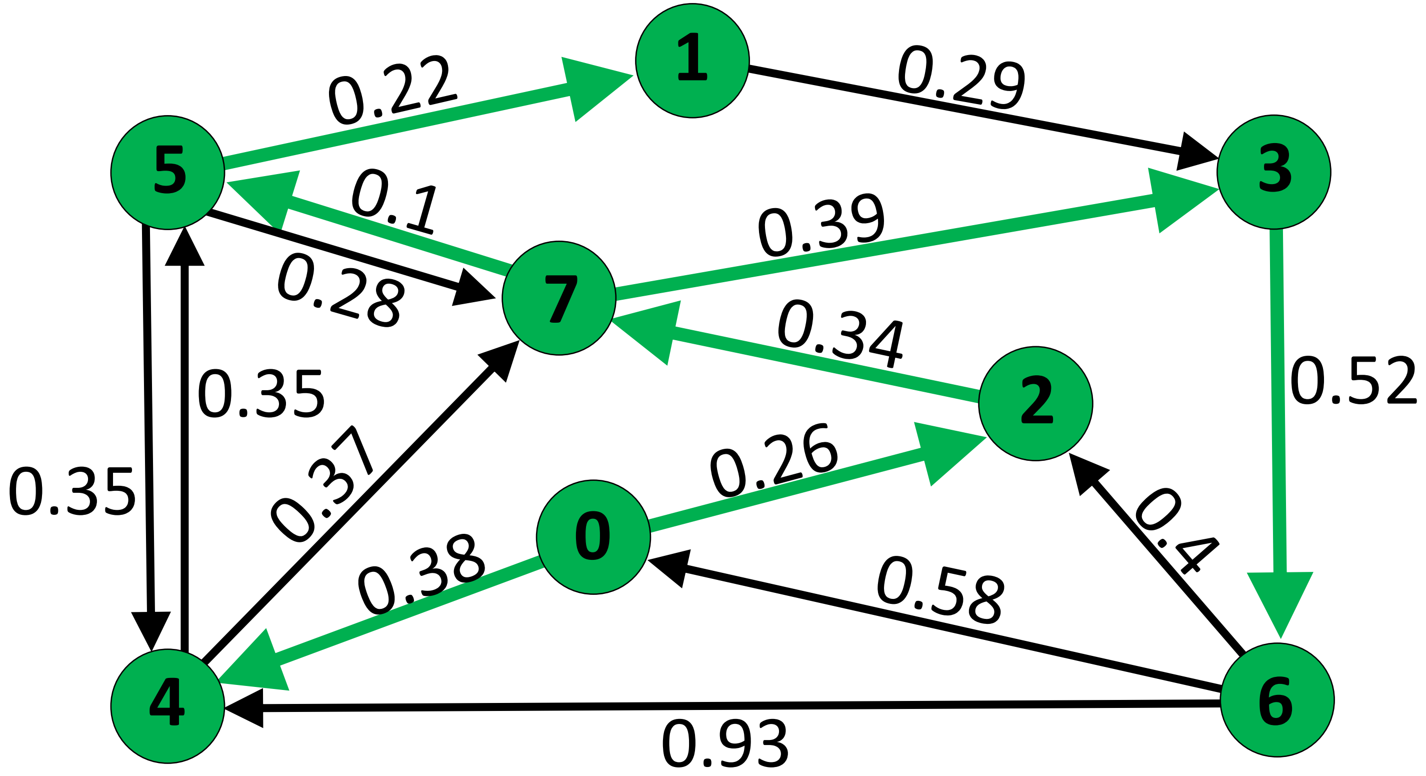
0	-
1	5
2	0
3	7
4	0
5	7
6	3
7	2

Priority queue

vertex (distance)

Shortest Path

queue top = 6 (1.51)



	Distance from 0
0	0
1	0.92
2	0.26
3	0.99
4	0.38
5	0.70
6	1.51
7	0.60

	Previous vertex
0	-
1	5
2	0
3	7
4	0
5	7
6	3
7	2

Priority queue

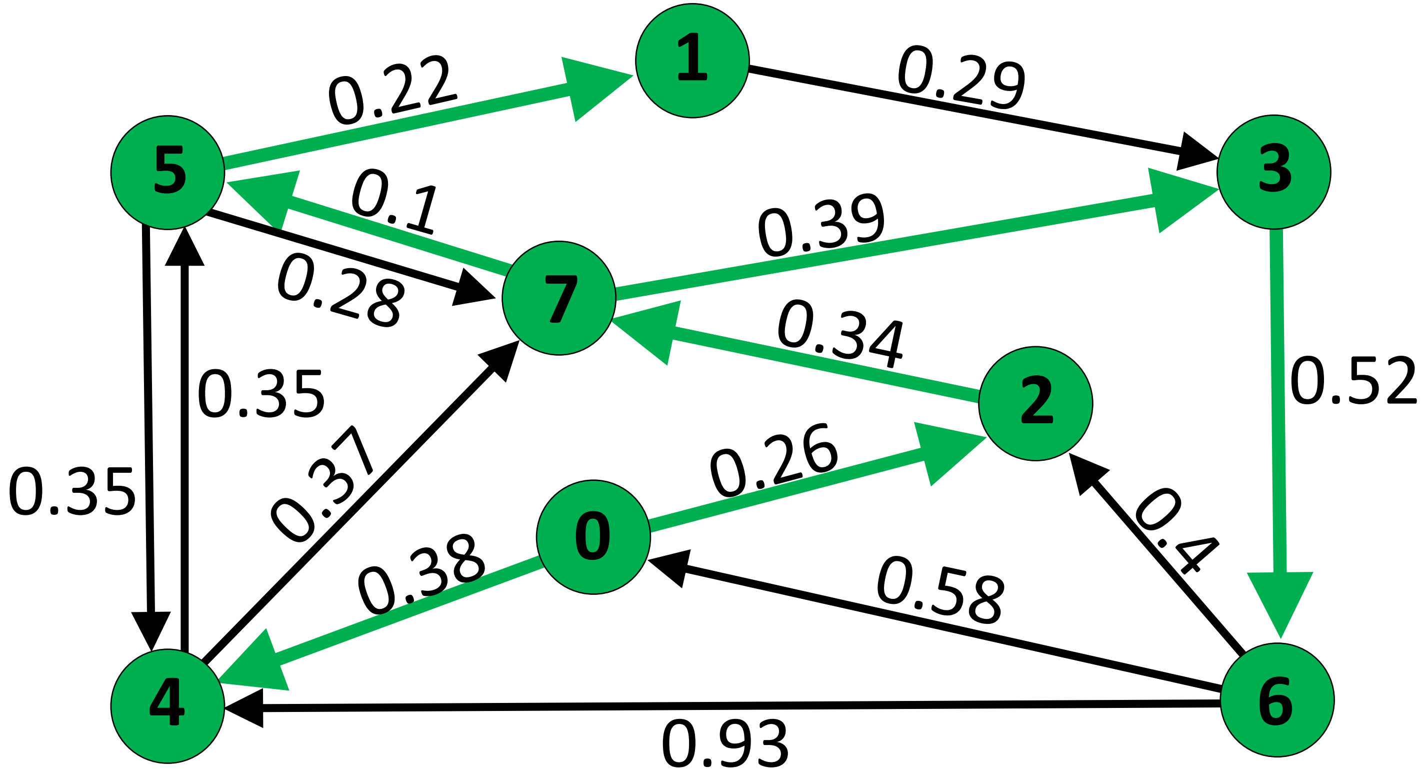
vertex (distance)

Repeat?

Neighbor 0? 1.51 + 0.58 > 0

Shortest Path

queue top = 6 (1.51)



Repeat?

Neighbor 2? 1.51 + 0.4 > 0.26

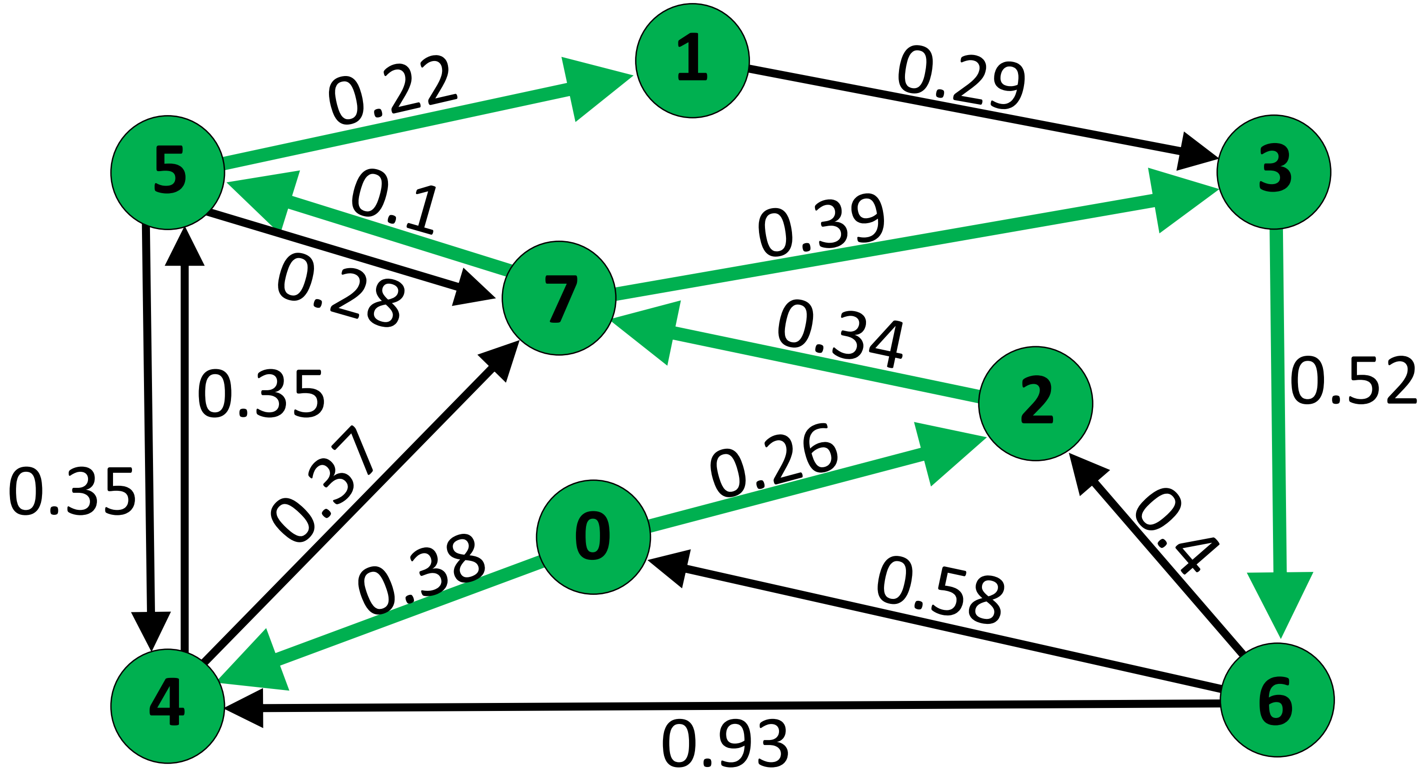
Vertex	Distance from 0	Previous vertex
0	0	-
1	0.92	5
2	0.26	0
3	0.99	7
4	0.38	0
5	0.70	7
6	1.51	3
7	0.60	2

Priority queue

vertex (distance)

Shortest Path

queue top = 6 (1.51)



When are we done?

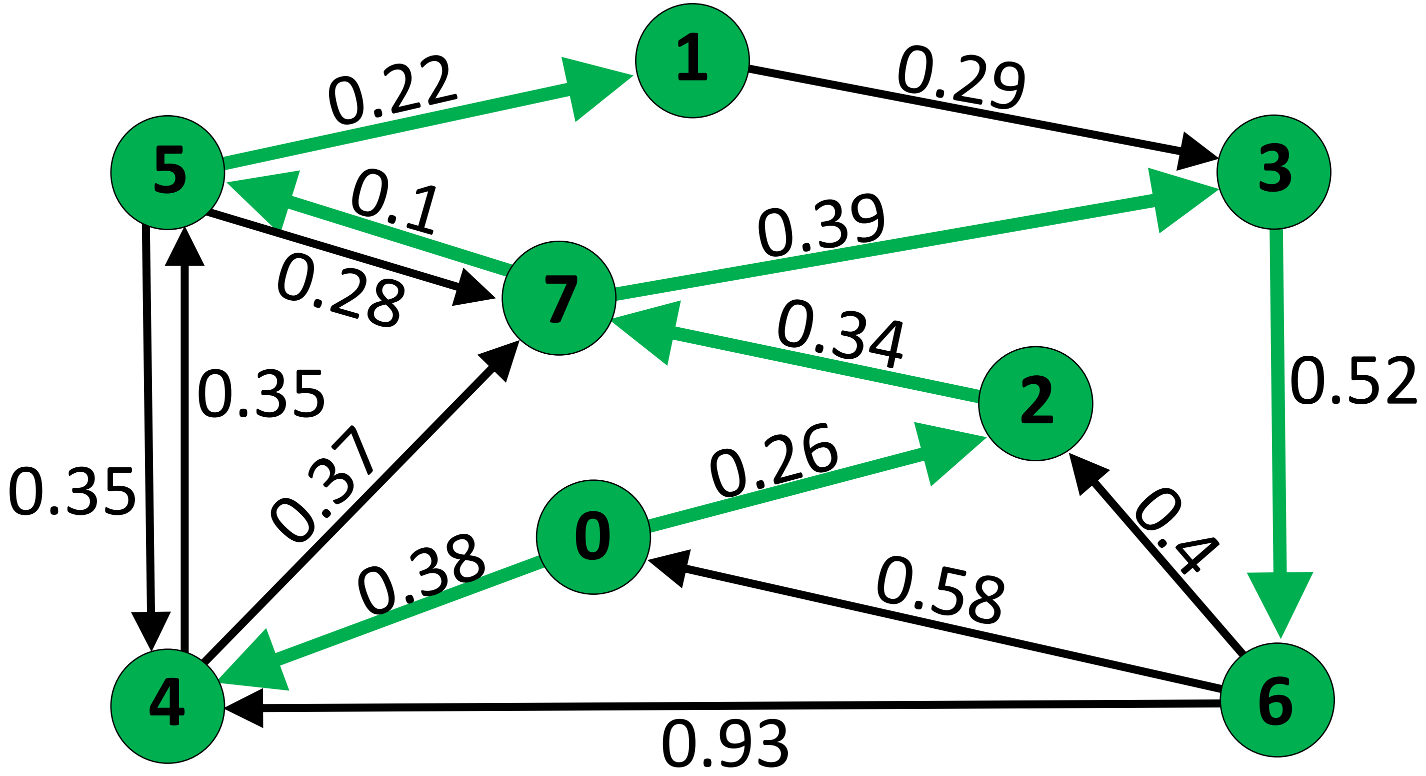
	Distance from 0	Previous vertex
0	0	-
1	0.92	5
2	0.26	0
3	0.99	7
4	0.38	0
5	0.70	7
6	1.51	3
7	0.60	2

Priority queue

vertex (distance)

Shortest Path

queue top = 6 (1.51)



Vertex	Distance from 0	Previous vertex
0	0	-
1	0.92	5
2	0.26	0
3	0.99	7
4	0.38	0
5	0.70	7
6	1.51	3
7	0.60	2

Priority queue

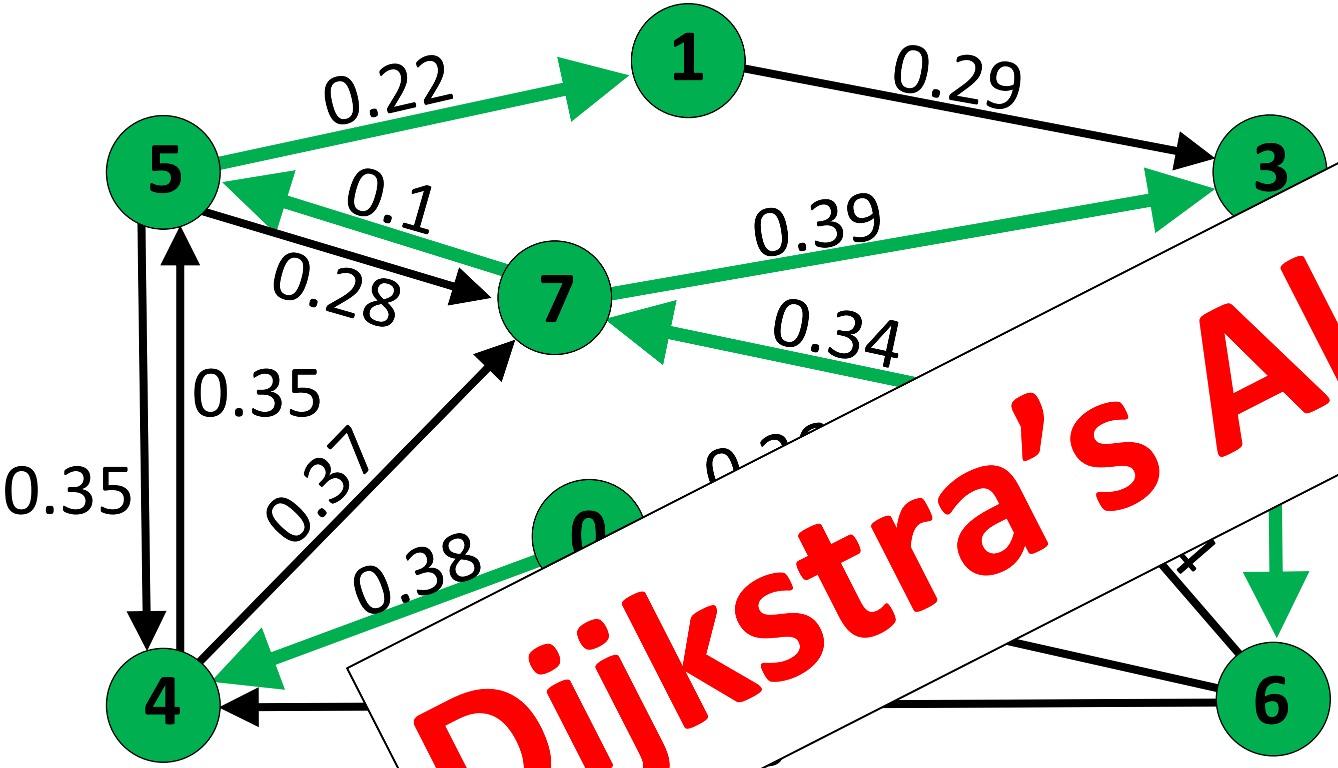
vertex (distance)

When are we done?

When the queue is empty!

Shortest Path

queue top = 6 (1.51)



	Distance from 0	Previous vertex
0	0	-
1	[illegible]	[illegible]
2	[illegible]	0
3	[illegible]	7
4	0.38	0
5	0.70	7
6	1.51	3
7	0.60	2

Priority queue

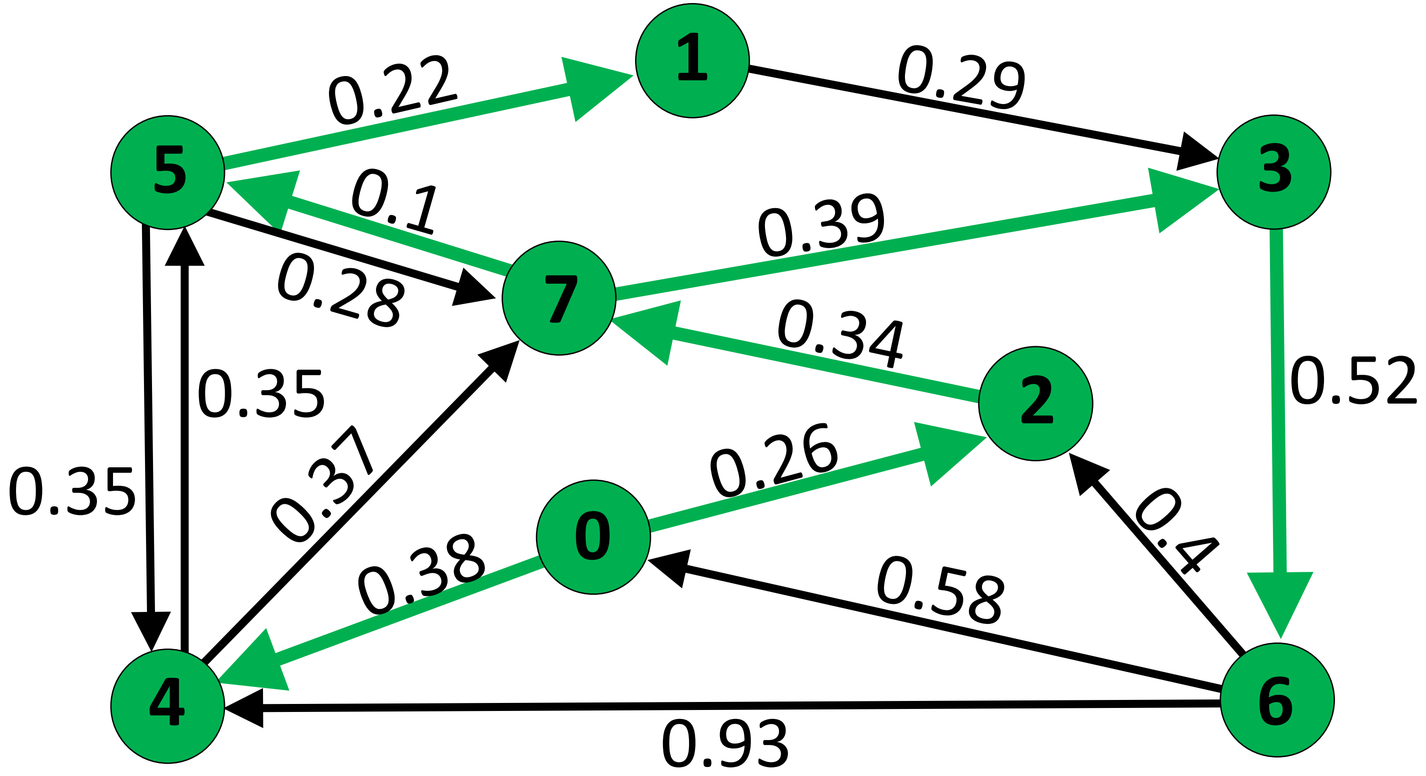
vertex (distance)

Dijkstra's Algorithm

When are we done?

When the queue is empty!

Shortest Path

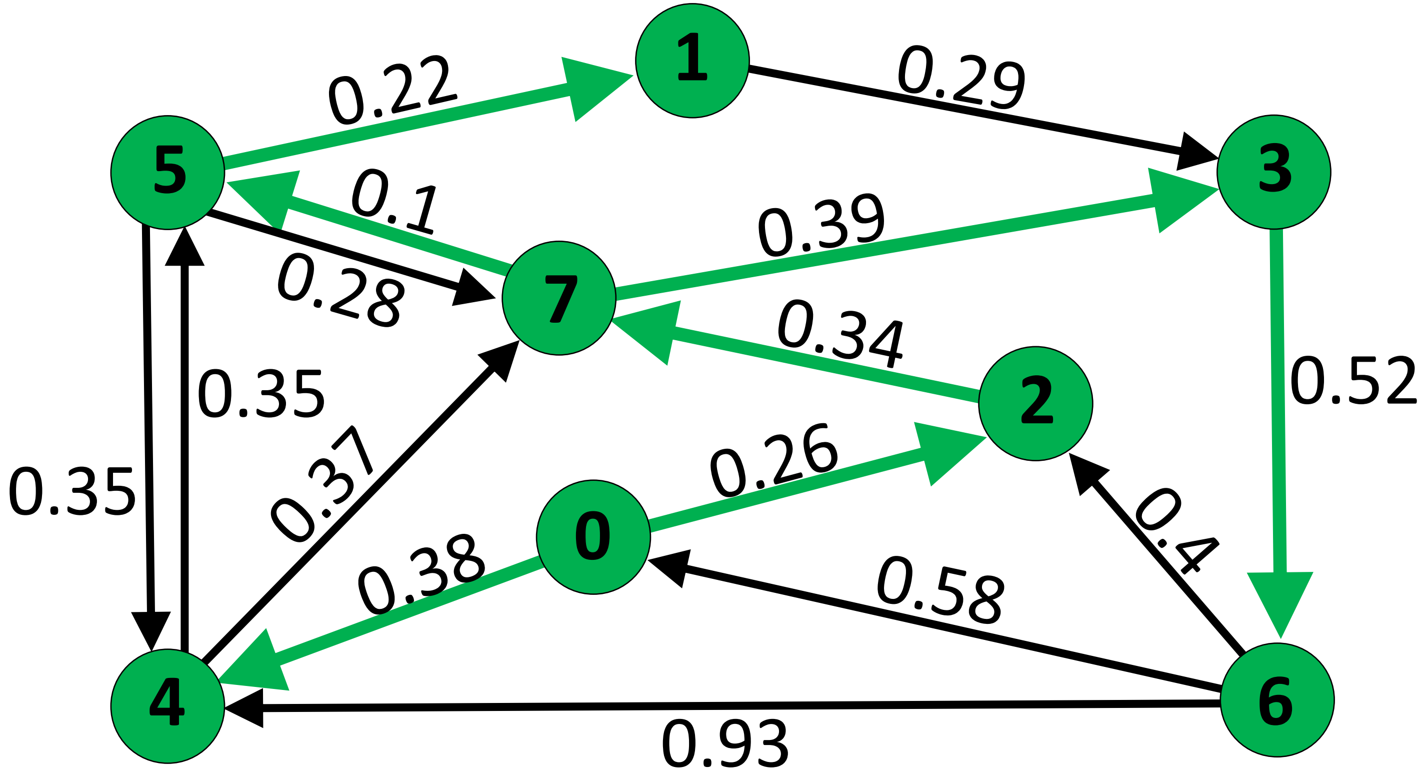


Assumptions:

- Graph is directed.
- Graph is edge-weighted.
- Edge weights are non-negative.
- Graph need not be simple (though our example will be).

What happens if there are self-loops?

Shortest Path



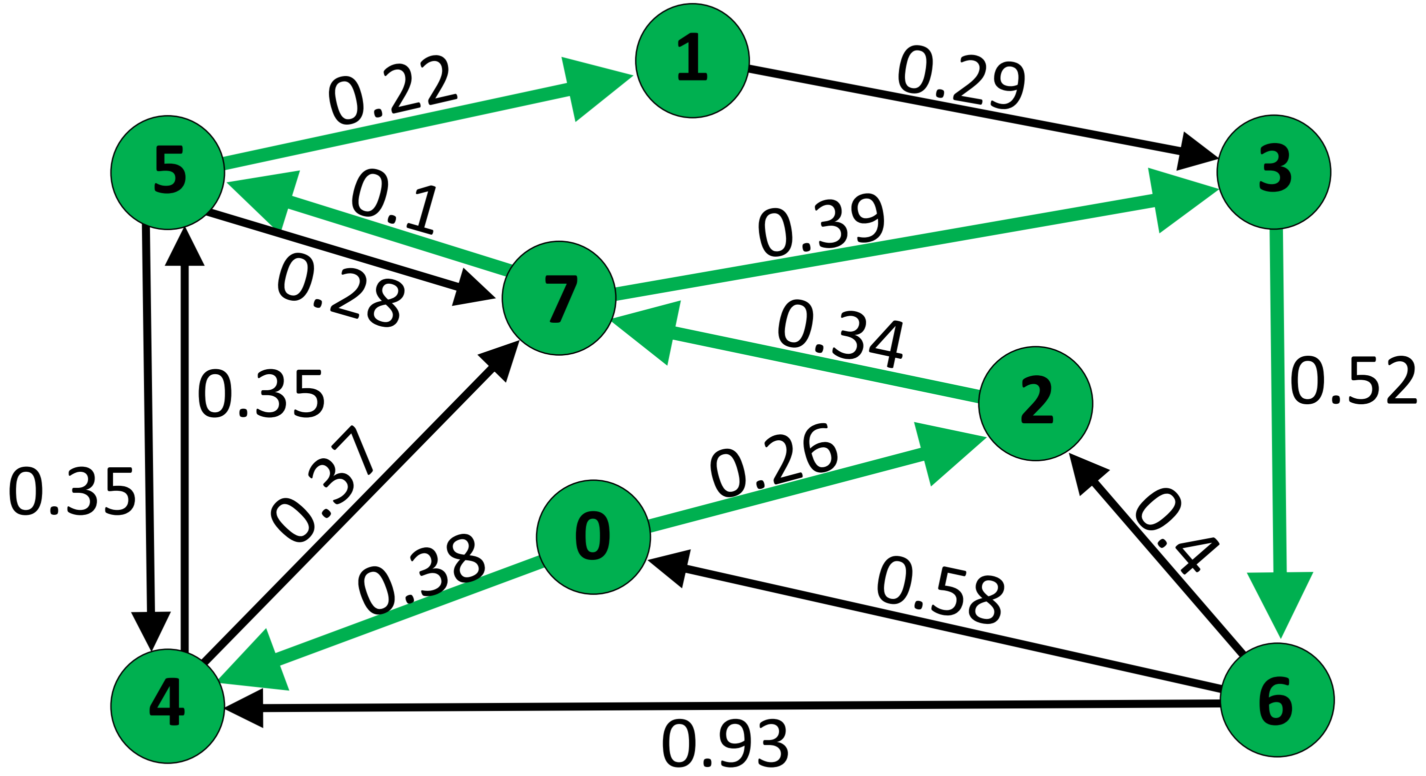
Assumptions:

- Graph is directed.
- Graph is edge-weighted.
- Edge weights are non-negative.
- Graph need not be simple (though our example will be).

What happens if there are self-loops?

They are never taken, since they will never lower the cost of a path.

Shortest Path

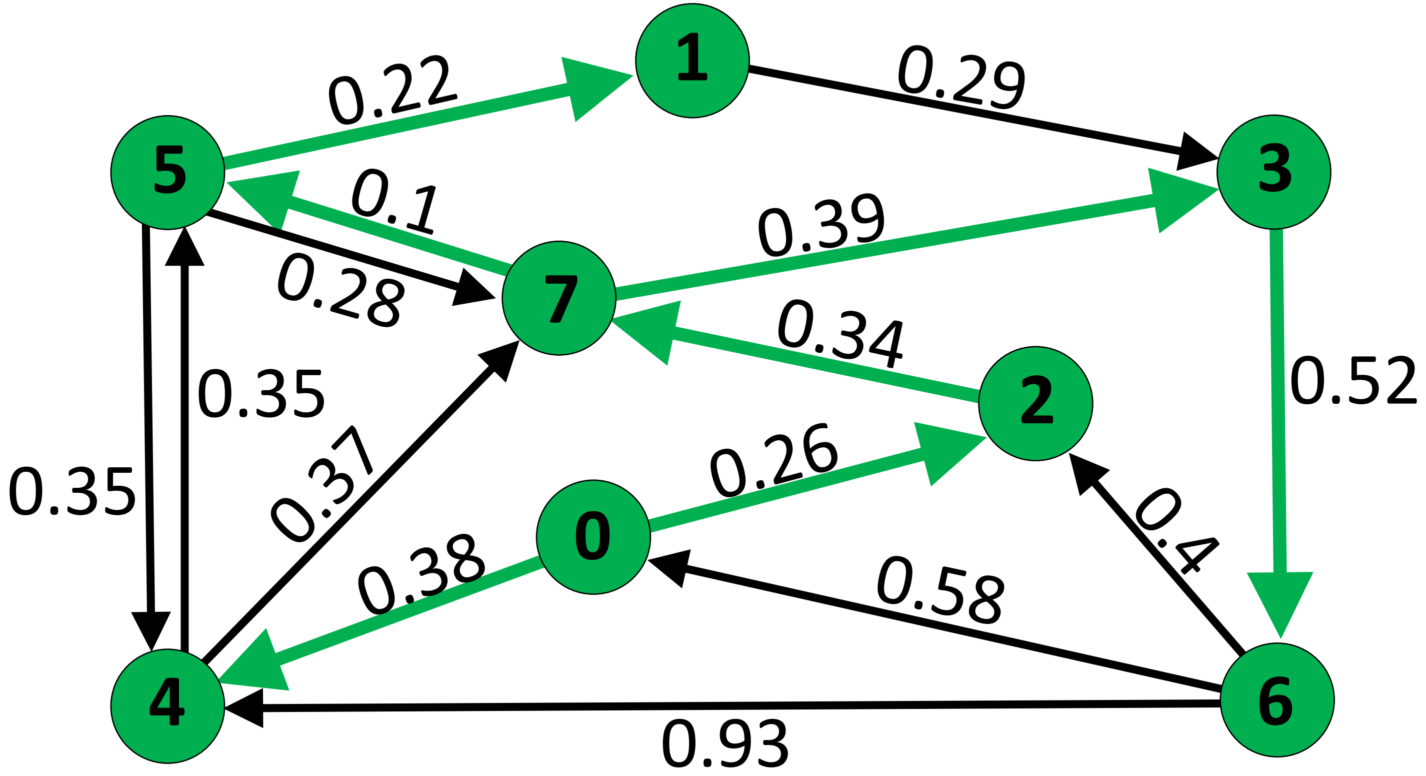


Assumptions:

- Graph is directed.
- Graph is edge-weighted.
- Edge weights are non-negative.
- Graph need not be simple (though our example will be).

What happens if there are parallel edges?

Shortest Path



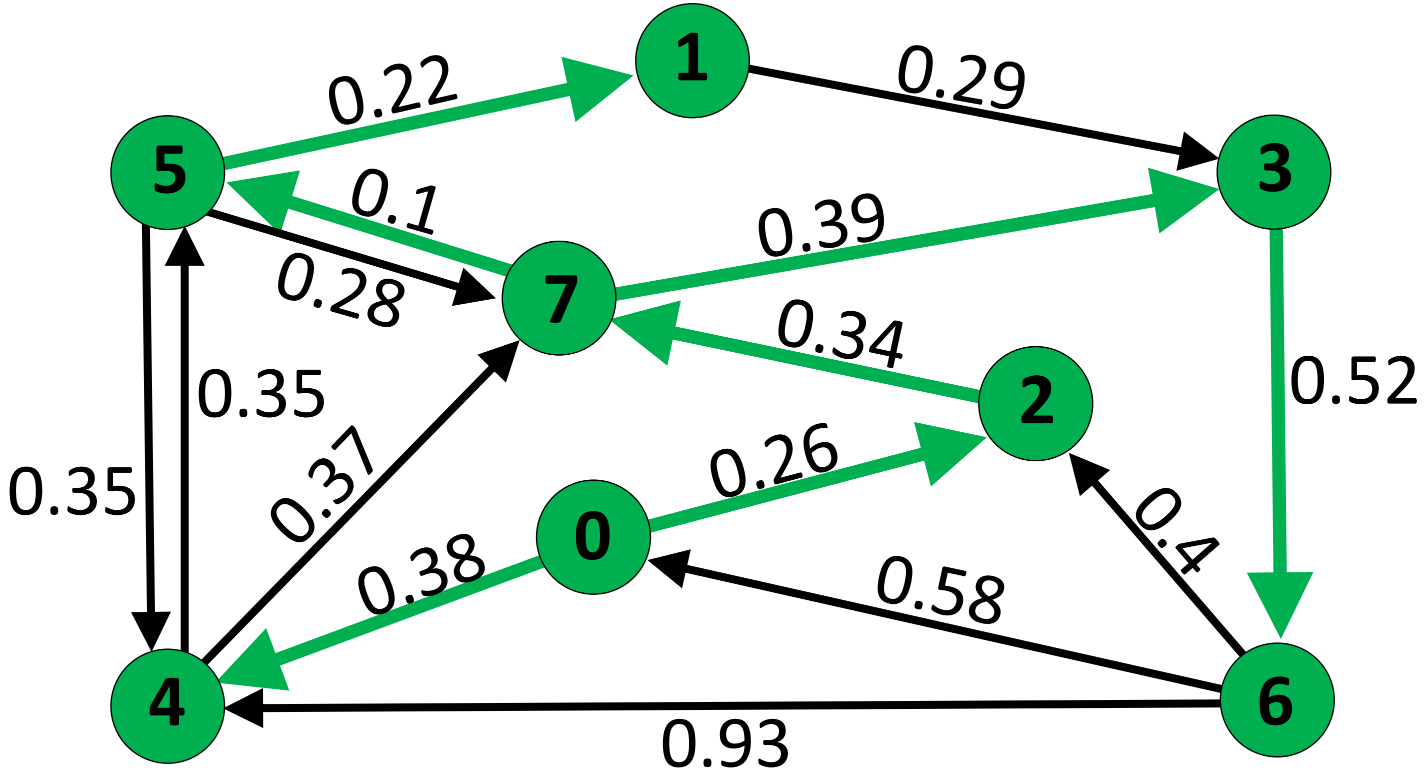
Assumptions:

- Graph is directed.
- Graph is edge-weighted.
- Edge weights are non-negative.
- Graph need not be simple (though our example will be).

What happens if there are parallel edges?

The cheapest one is taken and all others are ignored.

Shortest Path



Assumptions:

- Graph is directed.
- Graph is edge-weighted.
- Edge weights are non-negative.
- Graph need not be simple (though our example will be).

What happens if there are negative weights?

```
public class Edge implements Comparable<Edge>{

        private int sourceVertex;
        private int destVertex;
        private double weight;

        public Edge(int vertex1, int vertex2, double weight) {
                this.sourceVertex = vertex1;
                this.destVertex = vertex2;
                this.weight = weight;
        }
```