CSCI 232:
Data Structures and Algorithms

Dynamic Programming (Part 1)

Reese Pearsall

Spring 2025

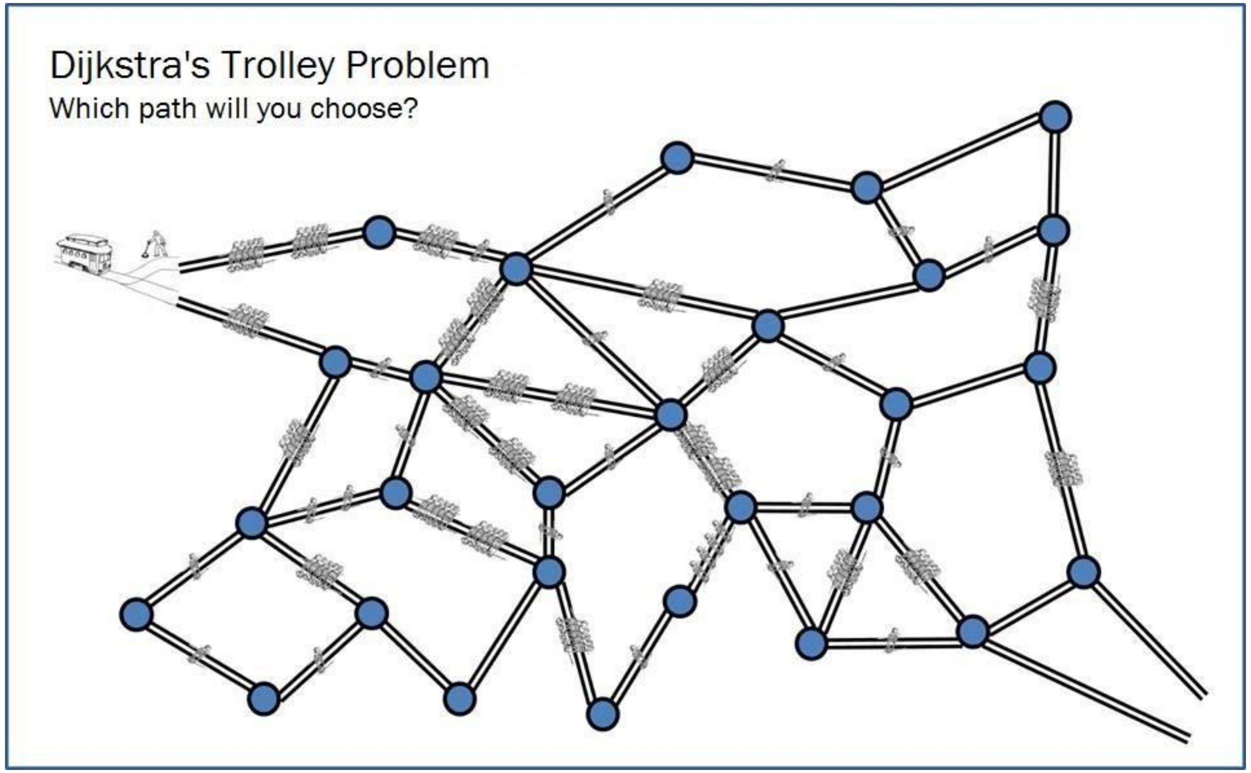
Announcements

Program 3 Due one week from today (4/22)

Tuesday April 22 will be an optional help session for Program 3 in Banard Hall 257 (no lecture)

Thursday's class will be a recorded lecture (no in-person class)

No lab this week



Change Making Problem

Given a set a coin denominations **D**, how can you represent **K** cents with the smallest number of coins?

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D = [1, 5, 10, 25]

K = 37

Change Making Problem

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D = [1, 5, 10, 25]

K = 37

Answer = 4

(Quarter, dime, two pennies)

Change Making Problem

Given a set a coin denominations **D**, how can you represent **K** cents with the smallest number of coins?

D = [1, 5, 10, 25]

K = 37

Answer = 4

(Quarter, dime, two pennies)

Algorithm?

Change Making Problem

Given a set a coin denominations **D**, how can you represent **K** cents with the smallest number of coins?

D = [1, 5, 10, 25]

K = 37

Answer = 4

(Quarter, dime, two pennies)

Use as many quarters as possible, then as many dimes as possible, …

Change Making Problem

Given a set a coin denominations **D**, how can you represent **K** cents with the smallest number of coins?

D = [1, 5, 10, 25]

K = 37

Answer = 4

(Quarter, dime, two pennies)

Use as many quarters as possible, then as many dimes as possible, …

This is known as the **greedy** approach

Change Making Problem

Given a set a coin denominations **D**, how can you represent **K** cents with the smallest number of coins?

D = [1, 5, 10, 25]

K = 37

Greedy Algorithm

Use as many quarters as possible, then as many dimes as possible, …

Change Making Problem

Given a set a coin denominations **D**, how can you represent **K** cents with the smallest number of coins?

D = [1, 5, 10, 18, 25]

K = 37

Greedy Algorithm

Use as many quarters as possible, then as many 18 cent pieces as possible, then dimes , …

What if there were also an 18-cent coin?

Change Making Problem

Given a set a coin denominations **D**, how can you represent **K** cents with the smallest number of coins?

D = [1, 5, 10, 18, 25]

K = 37

Greedy Algorithm

Use as many quarters as possible, then as many 18 cent pieces as possible, then dimes , …

25, 10, 1, 1 **(4 coins)**

What if there were also an 18-cent coin?

Change Making Problem

Given a set a coin denominations **D**, how can you represent **K** cents with the smallest number of coins?

D = [1, 5, 10, 18, 25]

K = 37

Greedy Algorithm

Use as many quarters as possible, then as many 18 cent pieces as possible, then dimes , …

25, 10, 1, 1 **(4 coins)**

What if there were also an 18-cent coin?

Real Answer = 18, 18, 1 (3 coins)

Change Making Problem

Given a set a coin denominations **D**, how can you represent **K** cents with the smallest number of coins?

D = [1, 5, 10, 18, 25]

K = 37

Greedy Algorithm

Use as many quarters as possible, then as many 18 cent pieces as possible, then dimes , …

25, 10, 1, 1 (4 coins)

What if there were also an 18-cent coin?

Real Answer = 18, 18, 1 (3 coins)

Lesson Learned: The Greedy approach works for the United States denominations, but not for a general set of denominations

Change Making Problem

Suppose I tell you that 2 quarters, 1 dime, and 3 pennies are the minimum number of coins needed to make **63 cents**

(We will assume we have the standard US denominations [1, 5, 10, 25] (NO 50 CENT PIECE)

$$25 + 25 + 10 + 1 + 1 + 1 = 63$$

What can you conclude?

Does this provide an answer to any other change making problems?

Change Making Problem

Suppose I tell you that 2 quarters, 1 dime, and 3 pennies are the minimum number of coins needed to make **63 cents**

(We will assume we have the standard US denominations [1, 5, 10, 25] (NO 50 CENT PIECE)

25 + 25 + 10 + 1 + 1 + 1 = 63

This is the minimum coins needed to make 38 cents

Change Making Problem

Suppose I tell you that 2 quarters, 1 dime, and 3 pennies are the minimum number of coins needed to make **63 cents**

(We will assume we have the standard US denominations [1, 5, 10, 25] (NO 50 CENT PIECE)

25 + 25 + 10 + 1 + 1 + 1 = 63

This is the minimum coins needed to make 13 cents

Change Making Problem

Suppose I tell you that 2 quarters, 1 dime, and 3 pennies are the minimum number of coins needed to make **63 cents**

(We will assume we have the standard US denominations [1, 5, 10, 25] (NO 50 CENT PIECE)

25 + 25 + 10 + 1 + 1 + 1 = 63

This is the minimum coins needed to make 3 cents

Change Making Problem

Suppose I tell you that 2 quarters, 1 dime, and 3 pennies are the minimum number of coins needed to make **63 cents**

(We will assume we have the standard US denominations [1, 5, 10, 25] (NO 50 CENT PIECE)

$$25 + 25 + 10 + 1 + 1 + 1 = 63$$

This is the minimum coins needed to make 2 cents

Change Making Problem

Suppose I tell you that 2 quarters, 1 dime, and 3 pennies are the minimum number of coins needed to make **63 cents**

(We will assume we have the standard US denominations [1, 5, 10, 25] (NO 50 CENT PIECE)

25 + 25 + 10 + 1 + 1 + 1 = 63

This is the minimum coins needed to make 1 cent

Change Making Problem

Suppose I tell you that 2 quarters, 1 dime, and 3 pennies are the minimum number of coins needed to make **63 cents**

(We will assume we have the standard US denominations [1, 5, 10, 25] (NO 50 CENT PIECE)

25 + 25 + 10 + 1 + 1 + 1 = 63

The solution to the change making problems consists of solutions to smaller change making problems

We can use **recursion** to solve this problem

In general, suppose a country has coins with denominations:

$1 = d_1 < d_2 < \cdots < d_k$ (US coins: $d_1 = 1, d_2 = 5, d_3 = 10, d_4 = 25$)

Algorithm: To make change for p cents, we are going to figure out change for every value $x < p$. We will build solution for p out of smaller solutions.

Change Making Problem

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. $0.25) of a coin used in the optimal solution.

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x – value (e.g. $0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 1 + C(12)

We used one quarter

Now find the minimum number of coins needed to make 12 cents

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. \$0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 1 + C(12)

C(12) = 1 + C(2)

We used one dime

Now find the minimum number of coins needed to make 2 cents

Change Making Problem

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. $0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 1 + C(12) 

C(12) = 1 + C(2) 

C(2) = 1 + C(1) 

C(1) = 1 + C(0) 

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. \$0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 1 + C(12) 

C(12) = 1 + C(2) 

C(2) = 1 + C(1) 

C(1) = 1

Change Making Problem

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. \$0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 1 + C(12) 

C(12) = 1 + C(2) 

C(2) = 1 + 1

Change Making Problem

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. \$0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 1 + C(12) 

C(12) = 1 + C(2) 

C(2) = 2

Change Making Problem

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. \$0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 1 + C(12) 

C(12) = 1 + 2 

Change Making Problem

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. \$0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 1 + C(12) 

C(12) = 3

Change Making Problem

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. $0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 1 + 3

Change Making Problem

$C(p)$ – minimum number of coins to make p cents.

x – value (e.g. $0.25) of a coin used in the optimal solution.

$$C(p) = 1 + C(p - x).$$

C(37) = 4

The minimum number of coins needed to make 37 cents is 4

In general, suppose a country has coins with denominations:

$$1 = d_1 < d_2 < \cdots < d_k$$

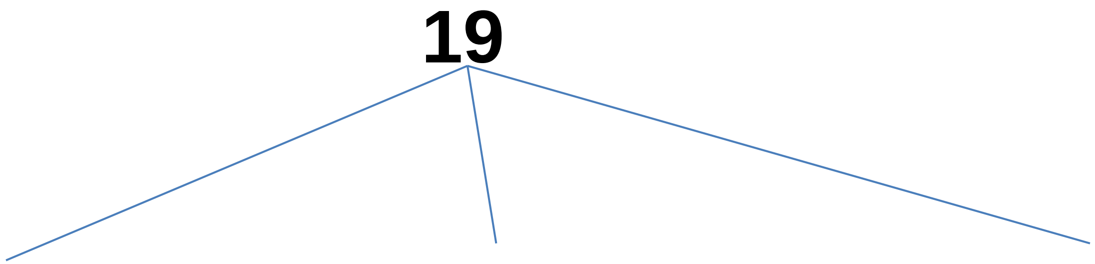
(US coins: $d_1 = 1, d_2 = 5, d_3 = 10, d_4 = 25$)

(This algorithm must work for ALL denominations)

Algorithm: To make change for p cents, we are going to figure out change for every value $x < p$. We will build solution for p out of smaller solutions.

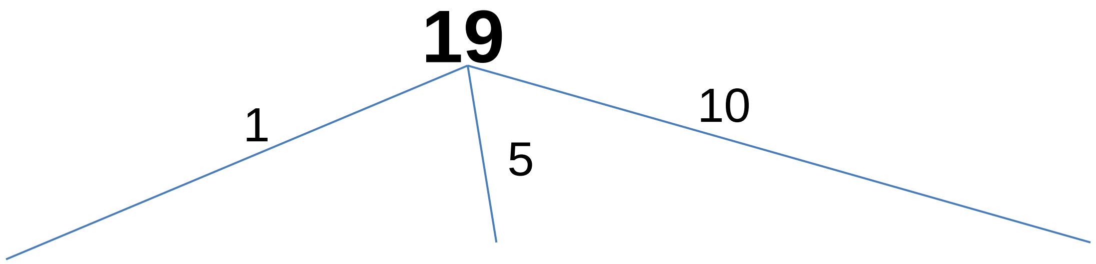
Change Making Problem

Make \$0.19 with \$0.01, \$0.05, \$0.10



Change Making Problem

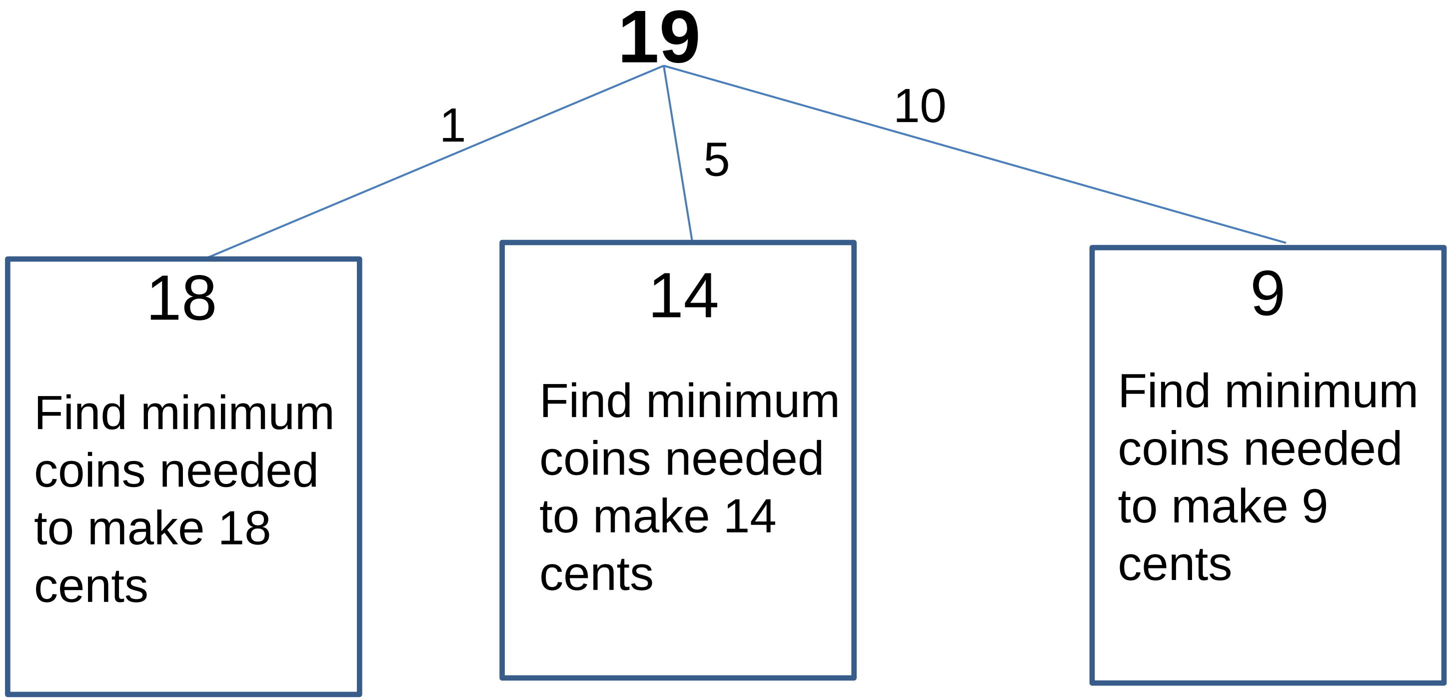
Make $0.19 with $0.01, $0.05, $0.10



Change Making Problem

Make \$0.19 with \$0.01, \$0.05, \$0.10

k = # denominations

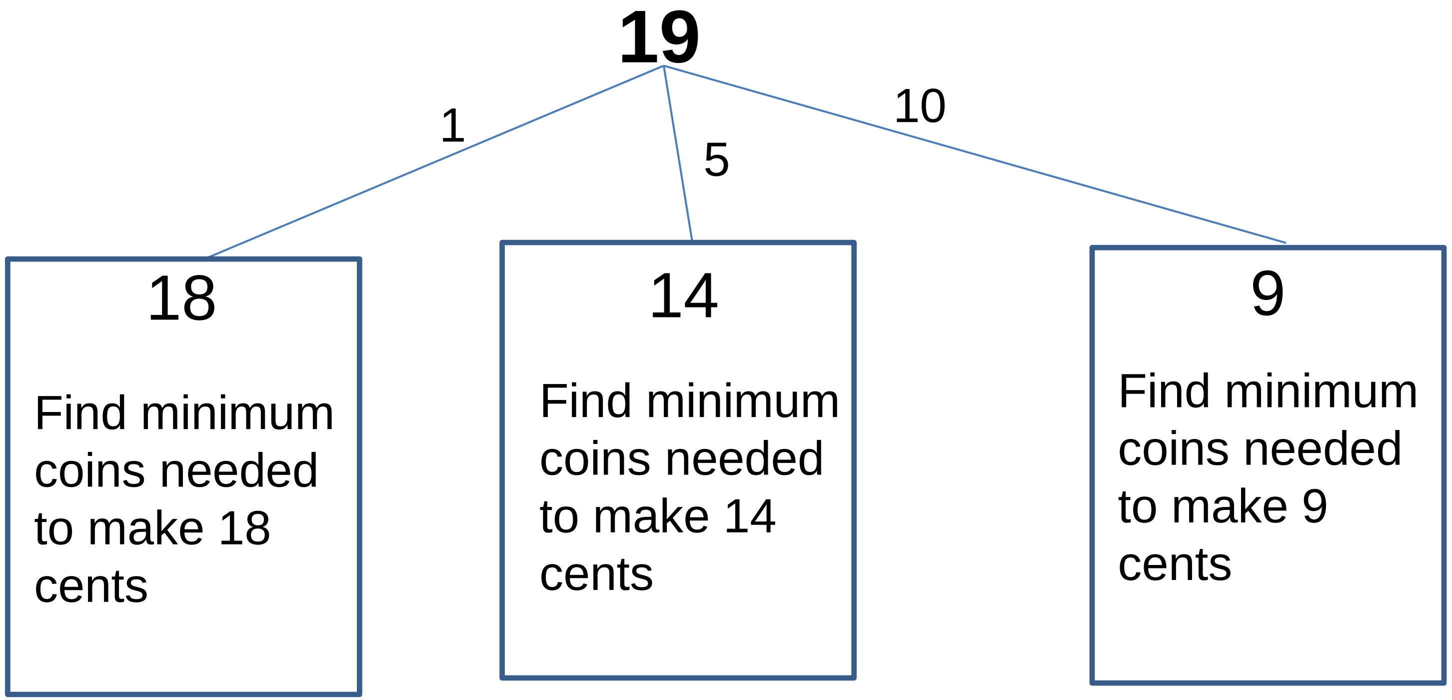


To find the minimum number of coins needed to create 19 cents, we generate **k** subproblems

Change Making Problem

Make \$0.19 with \$0.01, \$0.05, \$0.10

k = # denominations

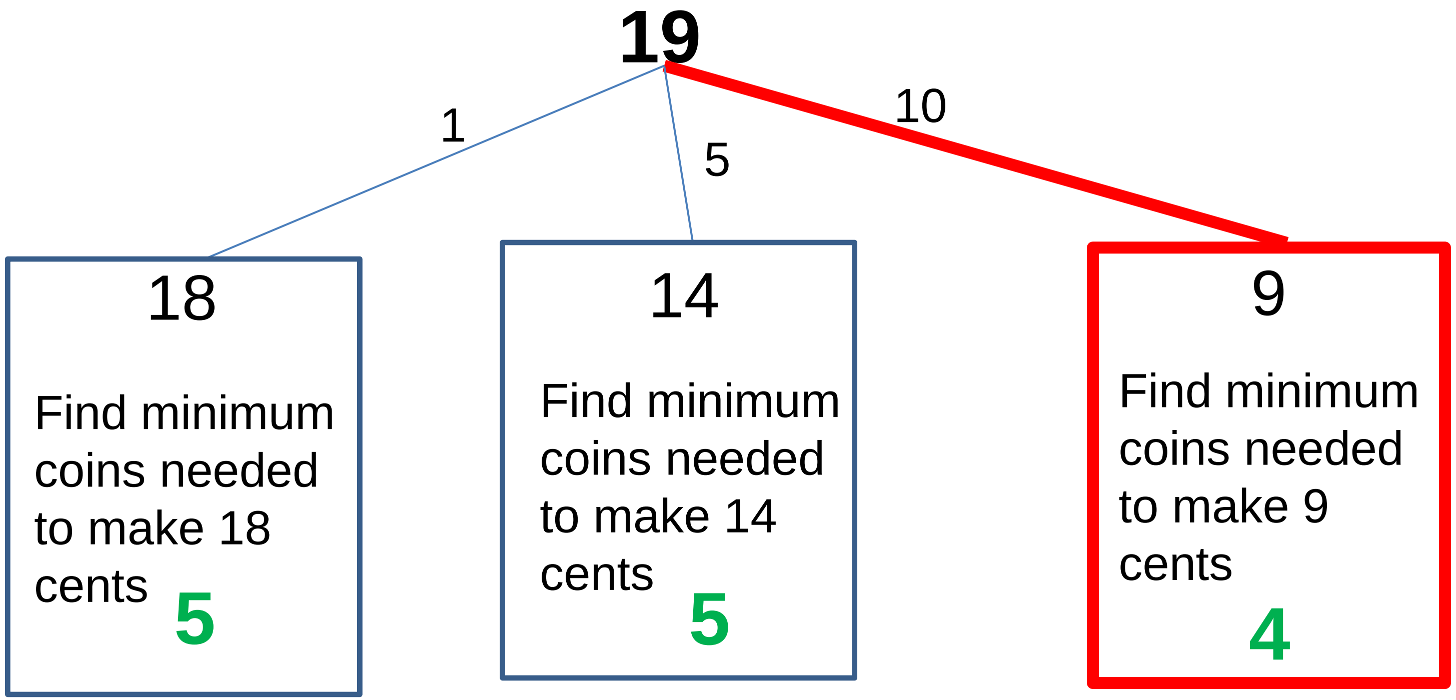


We want to select the **minimum** solution of these three subproblems

Change Making Problem

k = # denominations

Make \$0.19 with \$0.01, \$0.05, \$0.10

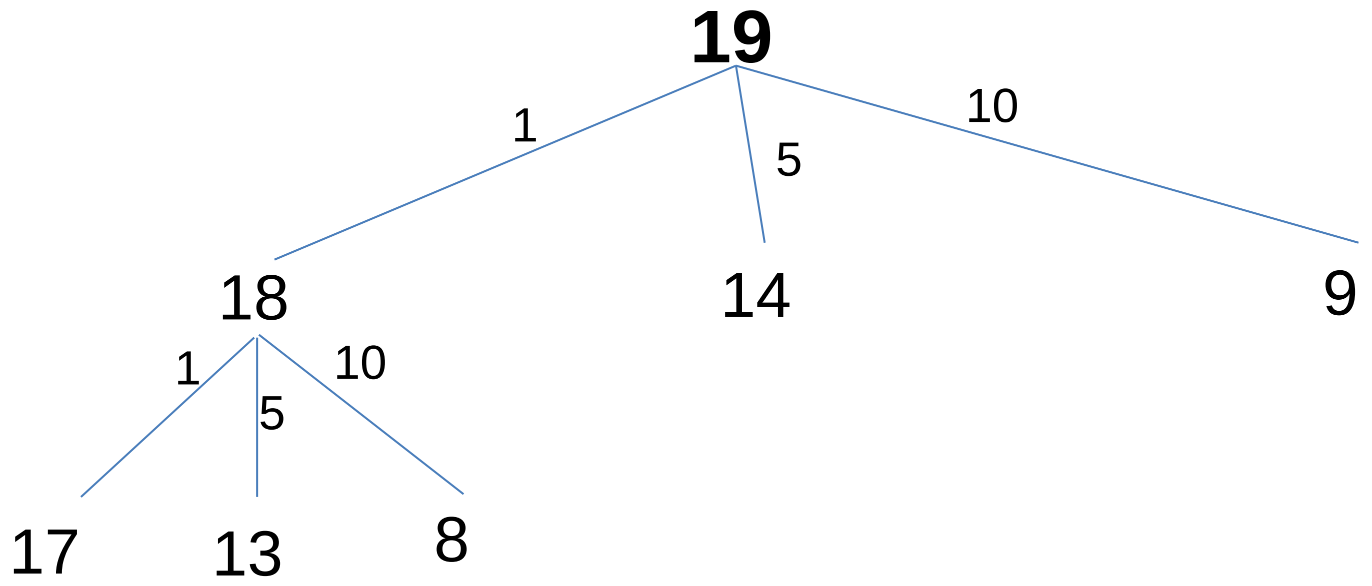


For the solution of our original problem (19), we want to select this branch (one dime used)

Change Making Problem

Make \$0.19 with \$0.01, \$0.05, \$0.10

k = # denominations



Find minimum coins needed to make 17 cents

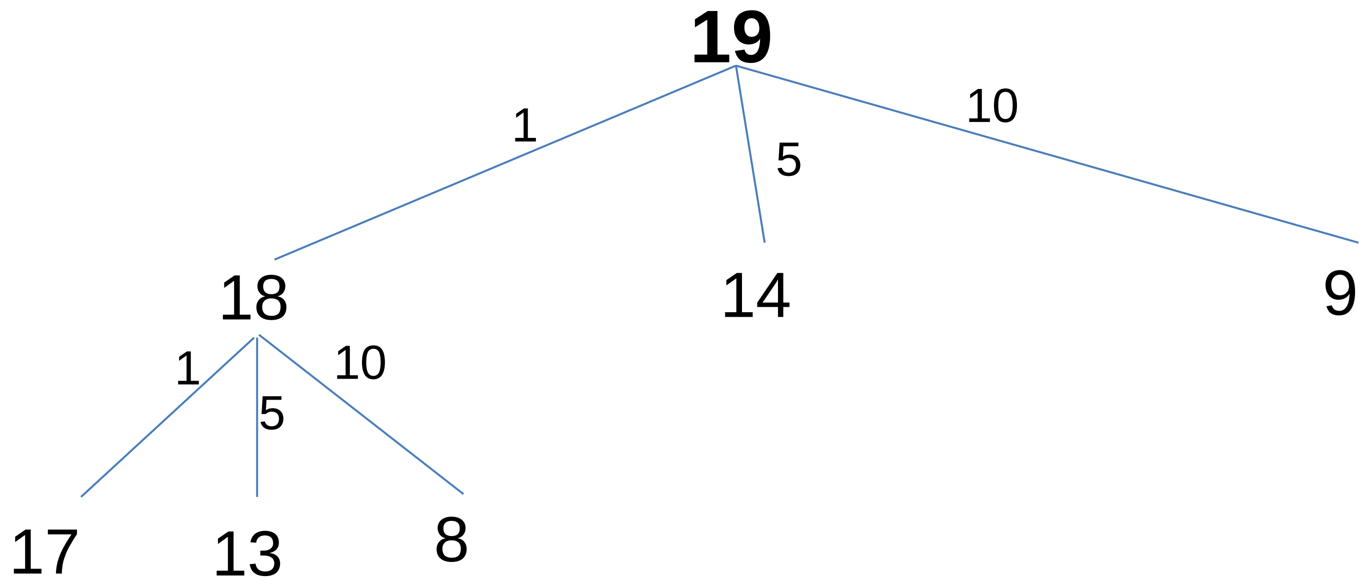
Find minimum coins needed to make 13 cents

Find minimum coins needed to make 8 cents

Change Making Problem

Make $0.19 with $0.01, $0.05, $0.10

k = # denominations



Change Making Problem

Make \$0.19 with \$0.01, \$0.05, \$0.10

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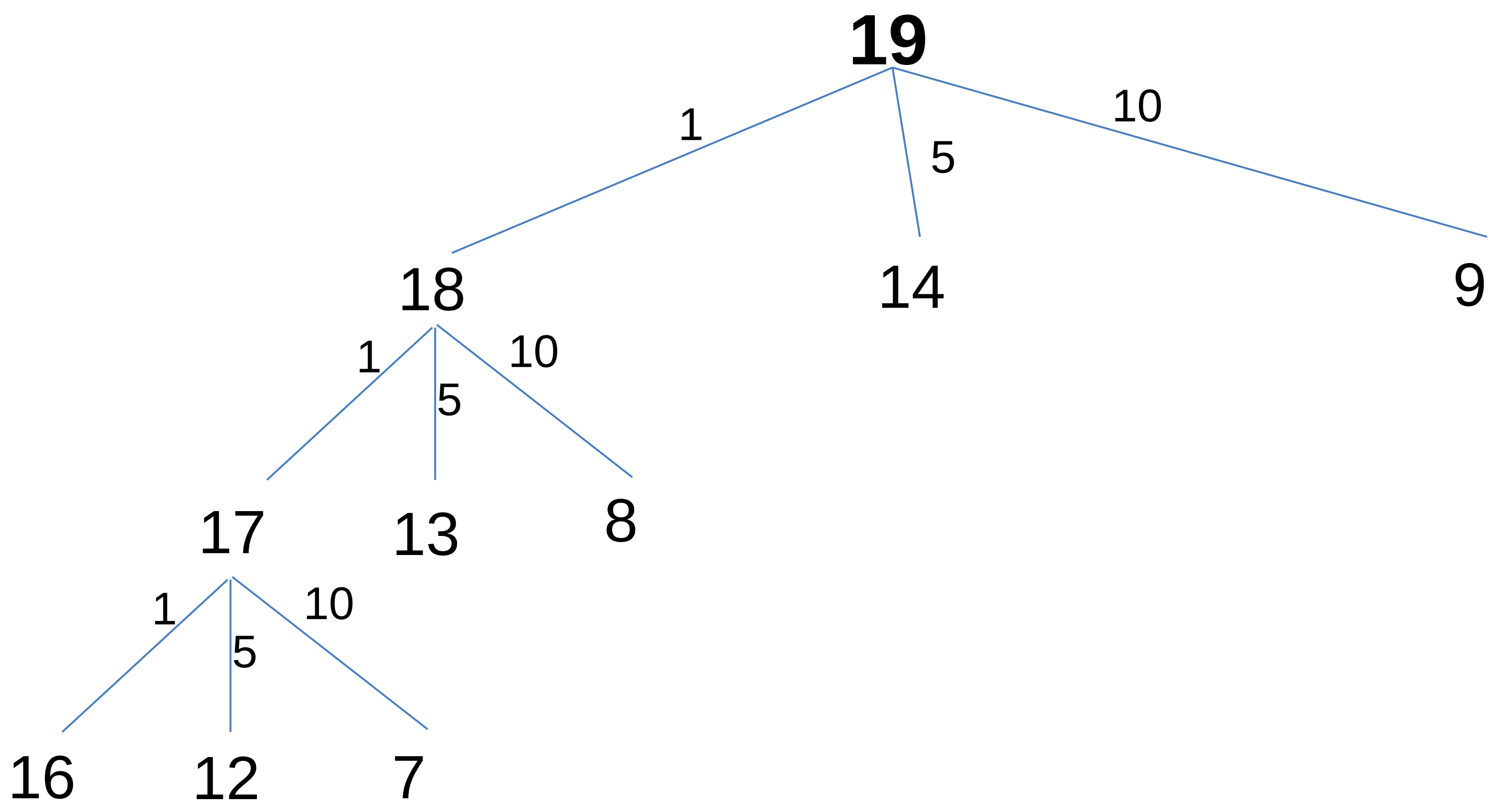

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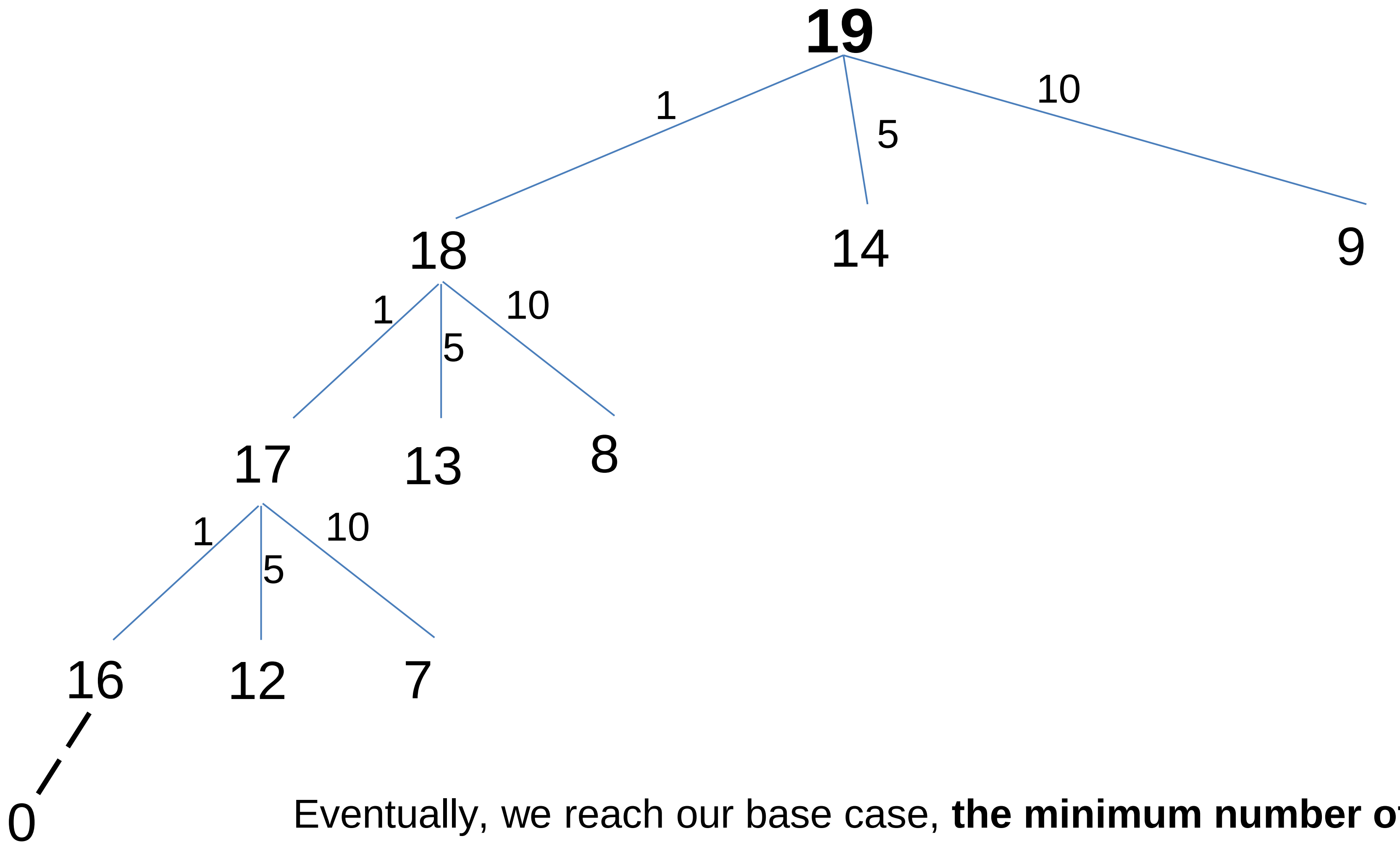
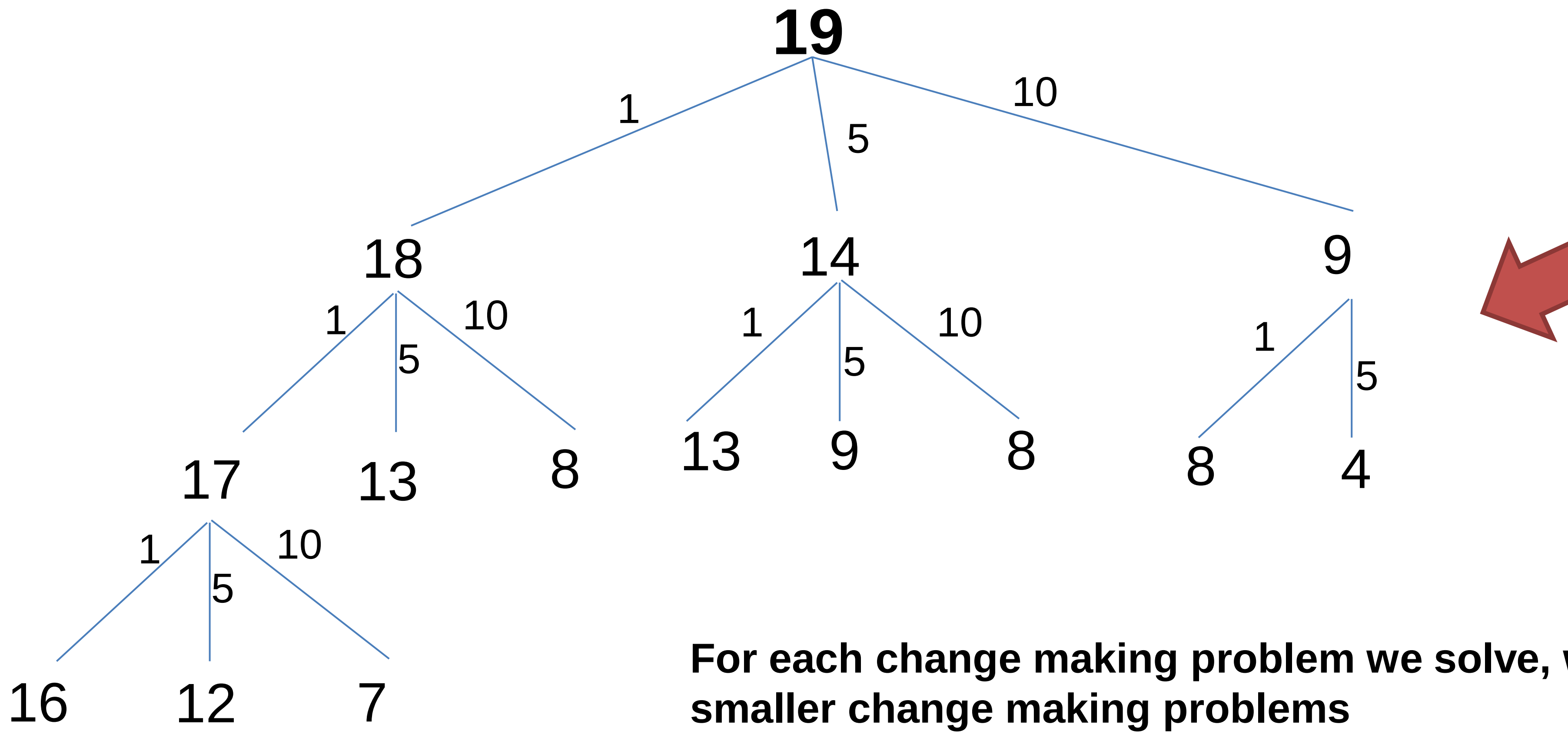


Eventually, we reach our base case, **the minimum number of coins needed to make 0 cents**

Change Making Problem

Make $0.19 with $0.01, $0.05, $0.10

k = # denominations



When C(9), we cant use a 10 cent piece…

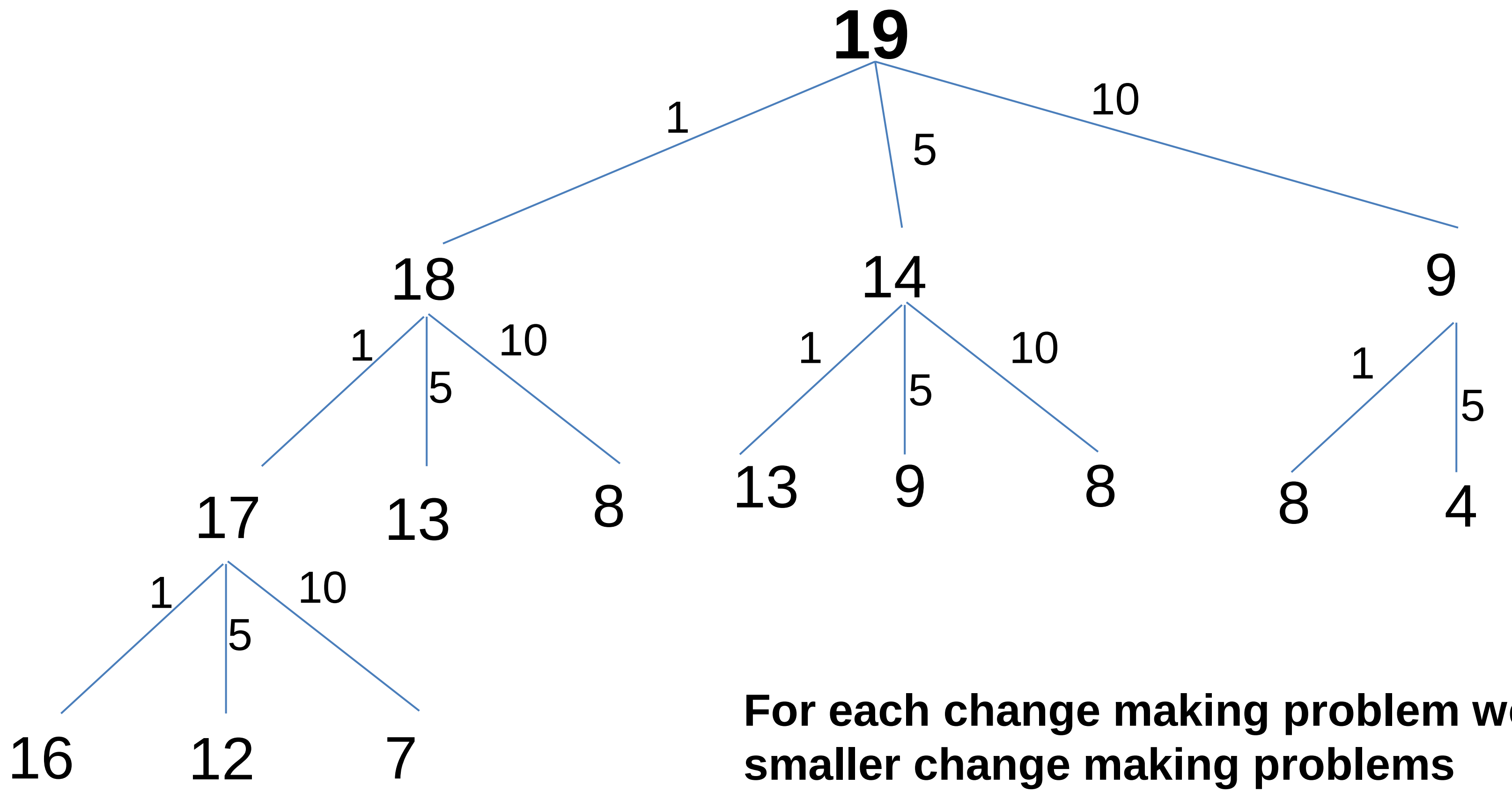
For each change making problem we solve, we must solve at most 3 smaller change making problems

Once we solve the smaller problems, we must select the branch that has the minimum value

Change Making Problem

Make $0.19 with $0.01, $0.05, $0.10

k = # denominations



When C(9), we cant use a 10 cent piece…

For each change making problem we solve, we must solve at most 3 smaller change making problems

Once we solve the smaller problems, we must select the branch that has the minimum value

Change Making Problem

$$C(p) = \begin{cases} \min_{i:d_i \le p} C(p - d_i) + 1, p > 0 \\ 0, p = 0 \end{cases}$$

Least change for 19 cents = minimum of:

- least change for 19-10 = 9 cents
- least change for 19-5 = 14 cents
- least change for 19-1 = 18 cents

For each problem P, we will solve the problem for (P – d), where d represents each possible denomination

Change Making Problem

$$C(p) = \begin{cases} \min_{i:d_i \le p} C(p - d_i) + 1, p > 0 \\ 0, p = 0 \end{cases}$$

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- least change for 19-10 = 9 cents
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For each problem P, we will solve the problem for (P – d), where d represents each possible denomination

We want to select only the branch the yields the minimum value

Change Making Problem

$$C(p) = \begin{cases} \min_{i:d_i \le p} C(p - d_i) + 1, p > 0 \\ 0, p = 0 \end{cases}$$

Least change for 19 cents = minimum of:

- least change for 19-10 = 9 cents
- least change for 19-5 = 14 cents
- least change for 19-1 = 18 cents

For each problem P, we will solve the problem for (P – d), where d represents each possible denomination

If we ever need to make change for 0 cents, return 0

We want to select only the branch the yields the minimum value

Change Making Problem

$$C(p) = \begin{cases} \min_{i:d_i \le p} C(p - d_i) + 1, p > 0 \\ 0, p = 0 \end{cases}$$

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If we ever need to make change for 0 cents, return 0

We want to select only the branch the yields the minimum value

Change Making Problem

D = array of denominations [1, 5, 10, 18, 25]
p = desired change (37)

```
min_coins(D, p)
```

Change Making Problem

D = array of denominations [1, 5, 10, 18, 25]
p = desired change (37)

```
min_coins(D, p)

   if p == 0
       return 0;
```

Base Case

Change Making Problem

D = array of denominations [1, 5, 10, 18, 25]
p = desired change (37)

```
min_coins(D, p)

    if p == 0
        return 0;

    else
        min = ∞
        a = ∞
```

Base Case

```
int min = Integer.MAX_VALUE;
int a = Integer.MAX_VALUE;;
```

Change Making Problem

D = array of denominations [1, 5, 10, 18, 25]
p = desired change (37)

```
min_coins(D, p)

   if p == 0
       return 0;

   else
      min = ∞
      a = ∞

      for each d_i in D
         if (p - d_i) >= 0
            a = min_coins(D, p - d_i )
```

Base Case

```java
int min = Integer.MAX_VALUE;
int a = Integer.MAX_VALUE;;
```

Recurse, and find the minimum number of coins needed using each valid denomination

Change Making Problem

D = array of denominations [1, 5, 10, 18, 25]
p = desired change (37)

```
min_coins(D, p)

  if p == 0
     return 0;

  else
     min = ∞
     a = ∞

     for each d_i in D
        if (p - d_i) >= 0
           a = min_coins(D, p - d_i )
        if a < min
           min = a
```

Base Case

```
int min = Integer.MAX_VALUE;
int a = Integer.MAX_VALUE;;
```

Recurse, and find the minimum number of coins needed using each valid denomination

Select the branch that has the minimum value

Change Making Problem

D = array of denominations [1, 5, 10, 18, 25]
p = desired change (37)

```
min_coins(D, p)
    if p == 0
        return 0;
    else
        min = ∞
        a = ∞

        for each d_i in D
            if (p - d_i) >= 0
                a = min_coins(D, p - d_i )
            if a < min
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Base Case

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int min = Integer.MAX_VALUE;
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Recurse, and find the minimum number of coins needed using each valid denomination

Select the branch that has the minimum value

Change Making Problem

D = array of denominations [1, 5, 10, 18, 25]
p = desired change (37)

```
min_coins(D, p)
    if p == 0
        return 0;
    else
        min = ∞
        a = ∞

        for each d_i in D
            if (p - d_i) >= 0
                a = min_coins(D, p - d_i )
            if a < min
                min = a

    return 1 + min
```

Base Case

```
int min = Integer.MAX_VALUE;
int a = Integer.MAX_VALUE;;
```

Recurse, and find the minimum number of coins needed using each valid denomination

Select the branch that has the minimum value

Once, our for loop finishes, we should know the branch that had the minimum, so return (1 + min), 1 because one coin was used in the current method call

Change Making Problem

```
min_coins(D, p)
    if p == 0
        return 0;
    else
        min = ∞
        a = ∞

        for each di in D
            if (p - di) >= 0
                a = min_coins(D, p - di )
            if a < min
                min = a

    return 1 + min
```

Change Making Problem

```
min_coins(D, p)
    if p == 0
        return 0;
    else
        min = ∞
        a = ∞

        for each di in D
            if (p - di) >= 0
                a = min_coins(D, p - di )
            if a < min
                min = a

        return 1 + min
```

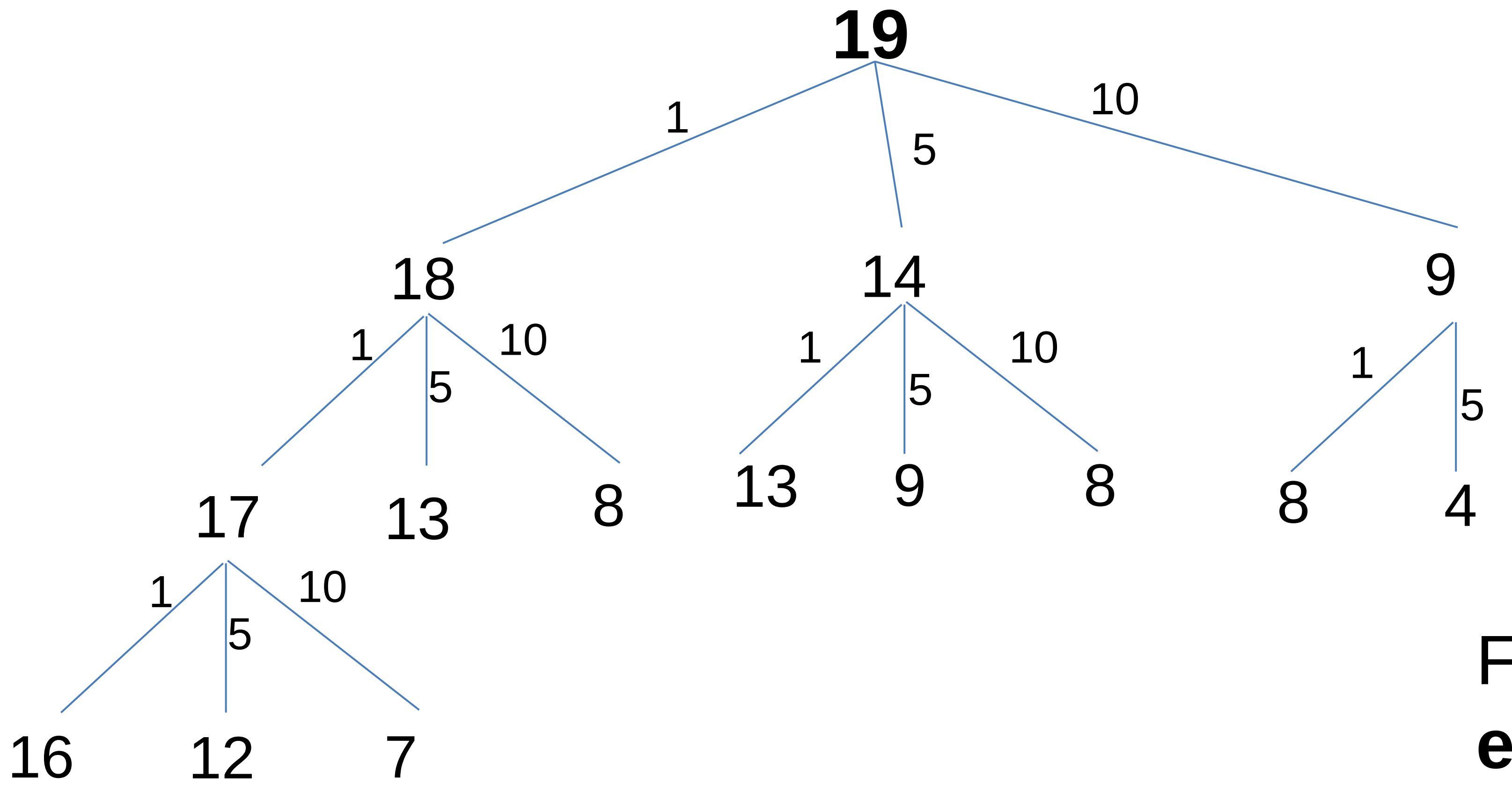
Running time?

Change Making Problem

Make $0.19 with $0.01, $0.05, $0.10

k = # denominations

p = value to make change for



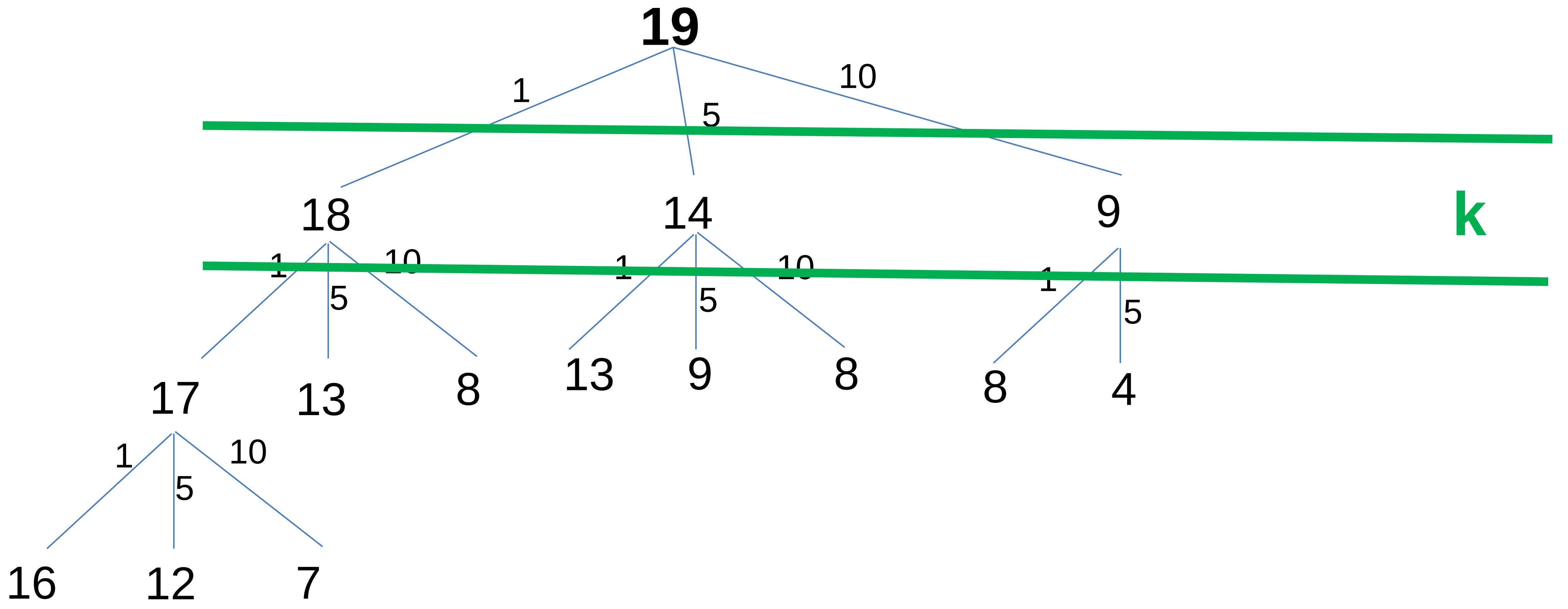
For sufficiently large p, **every permutation** of denominations is included.

Change Making Problem

Make \$0.19 with \$0.01, \$0.05, \$0.10

k = # denominations

p = value to make change for



Change Making Problem

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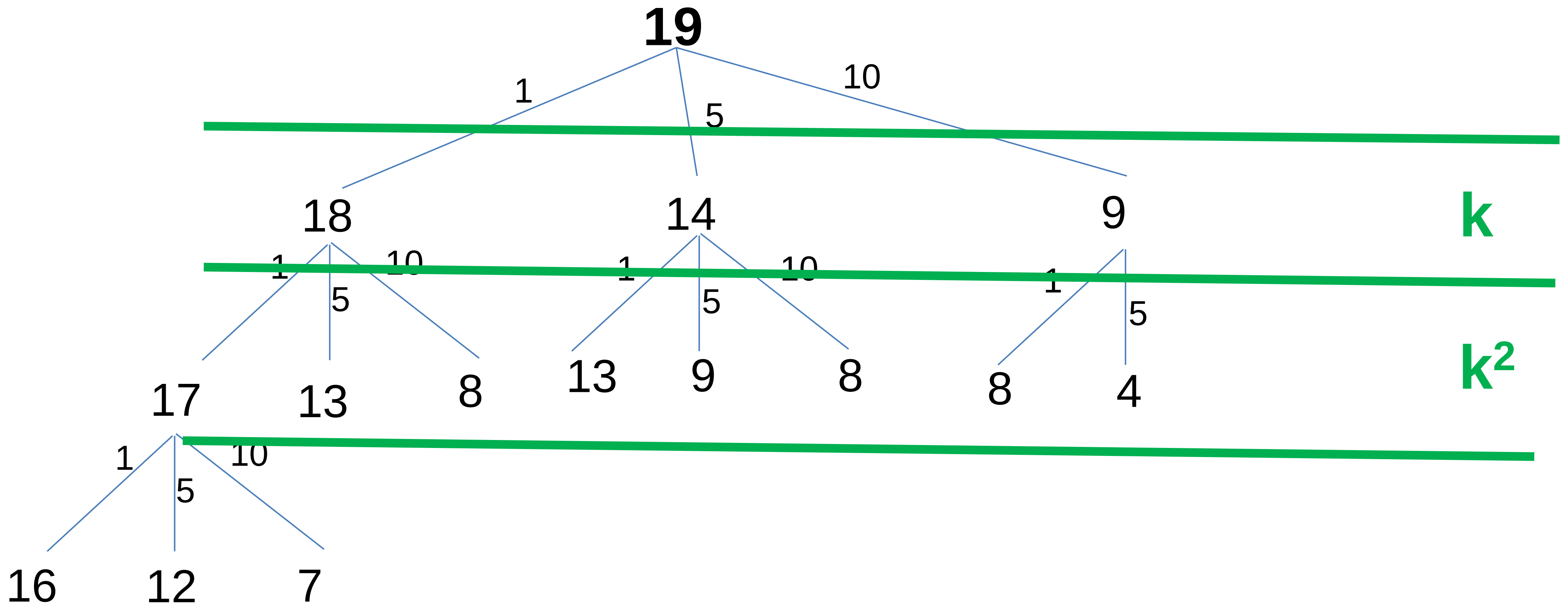

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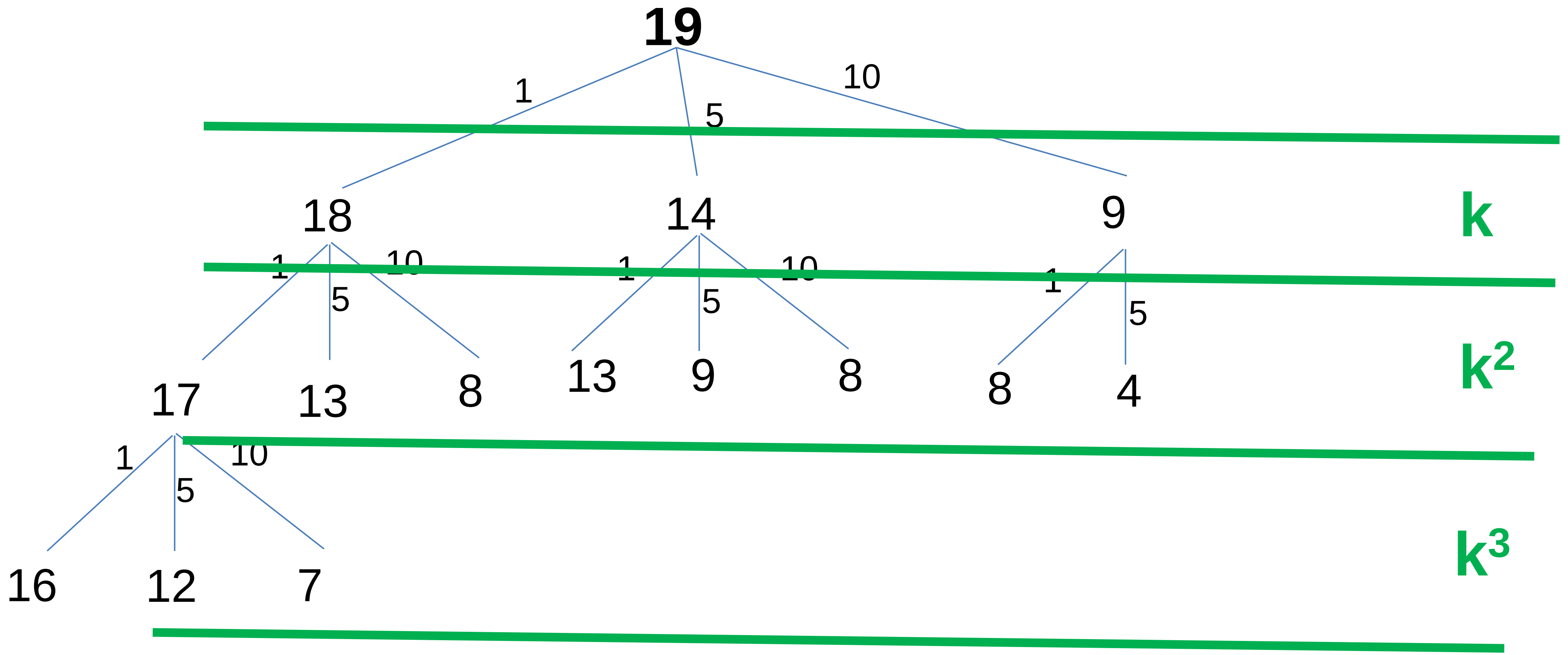

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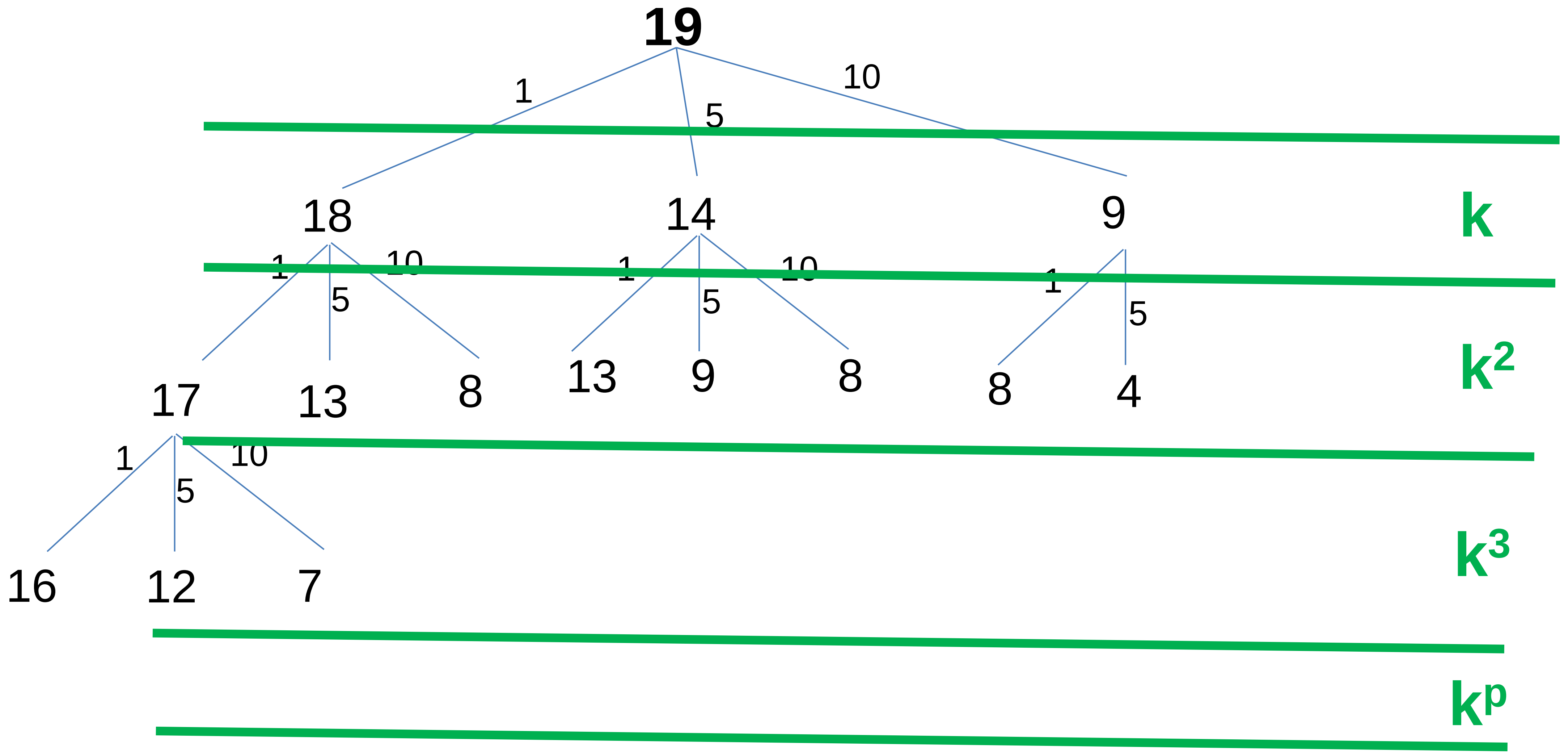


Change Making Problem

Make \$0.19 with \$0.01, \$0.05, \$0.10

k = # denominations

p = value to make change for

19

As k and p both grow, the number of recursive calls being made will grow **exponentially**

If we have a lot of coin denominations, we will have **a lot** of branching

k

k^2

k^3

k^p

Change Making Problem

Make \$0.19 with \$0.01, \$0.05, \$0.10

k = # denominations

p = value to make change for

19

As k and p both grow, the number of recursive calls being made will grow **exponentially**

Running time: O(💀)

k

k^2

k^3

k^p

Change Making Problem

Make \$0.19 with \$0.01, \$0.05, \$0.10

k = # denominations

p = value to make change for

19

As k and p both grow, the number of recursive calls being made will grow **exponentially**

Running time: probably $O(k^p)$ or $O(k!)$

For a large set of denominations, or a large p, this algorithm will take a long time to run

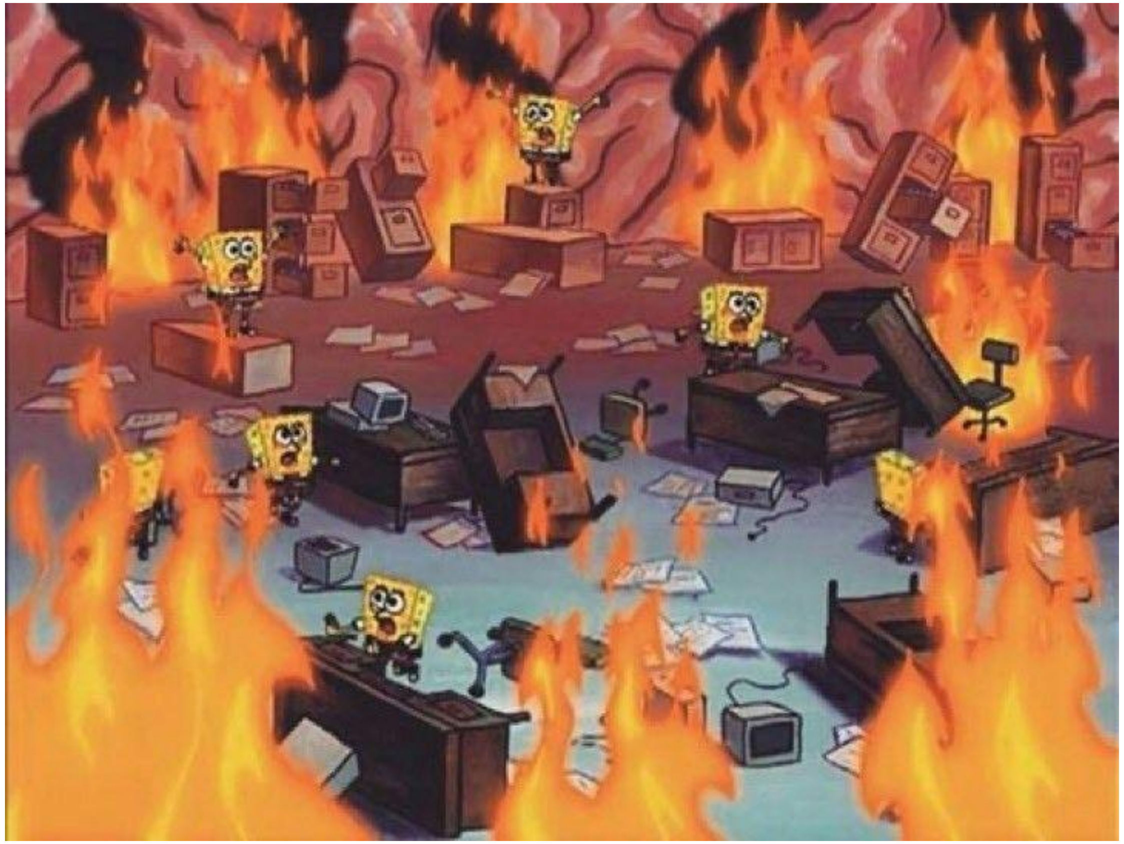
k

k^2

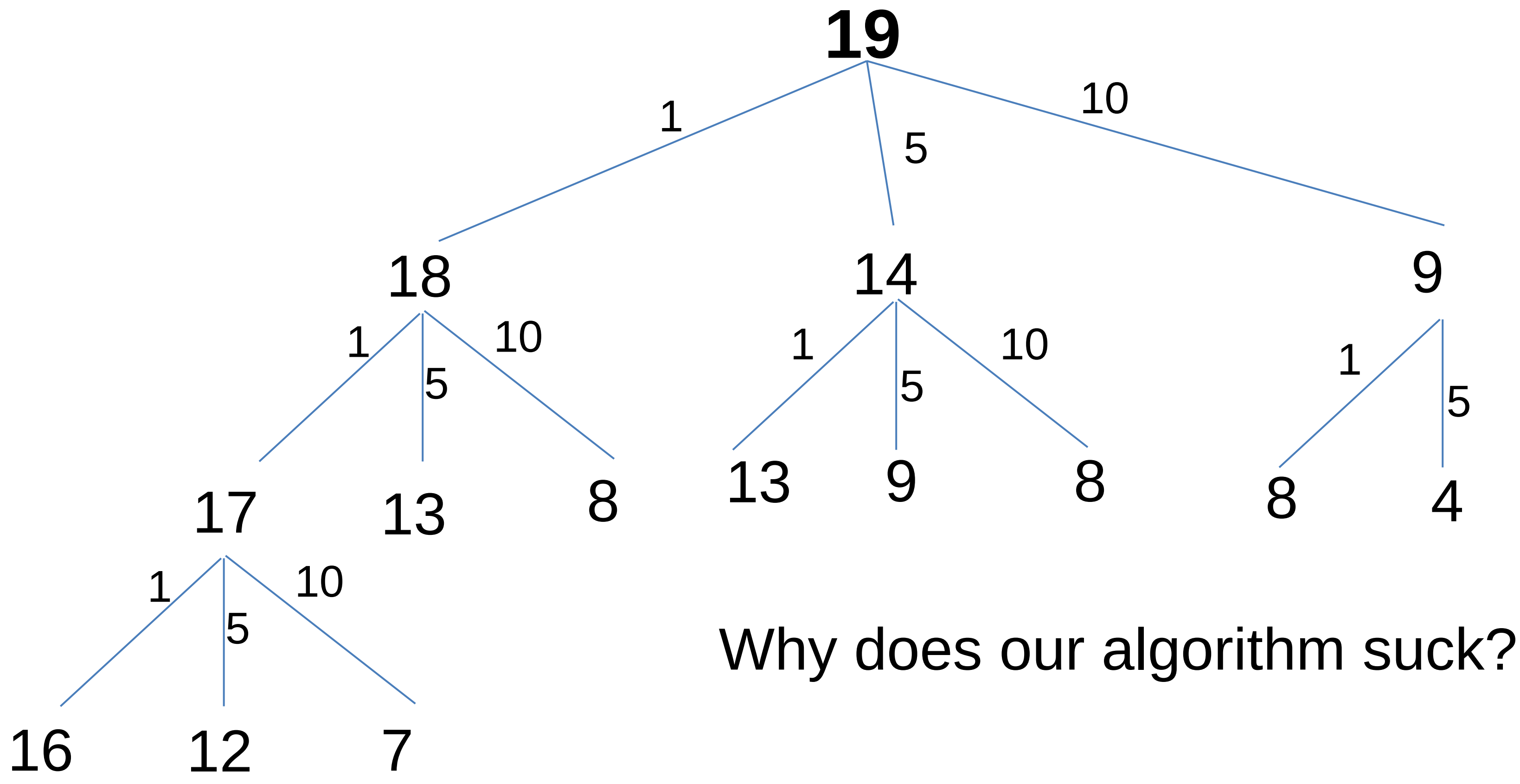
k^3

k^p

Let's try 81 cents!

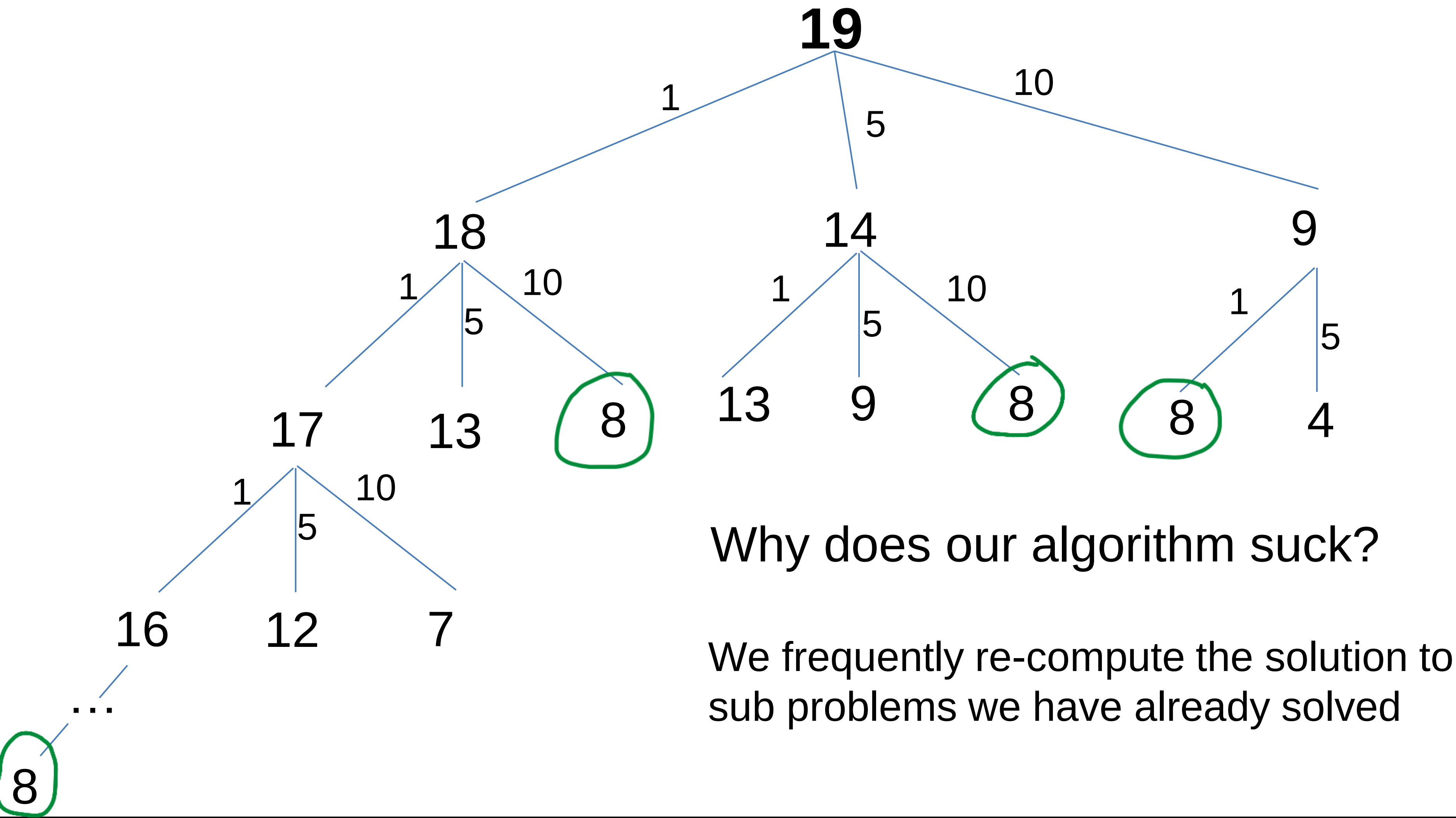


Change Making Problem



Why does our algorithm suck?

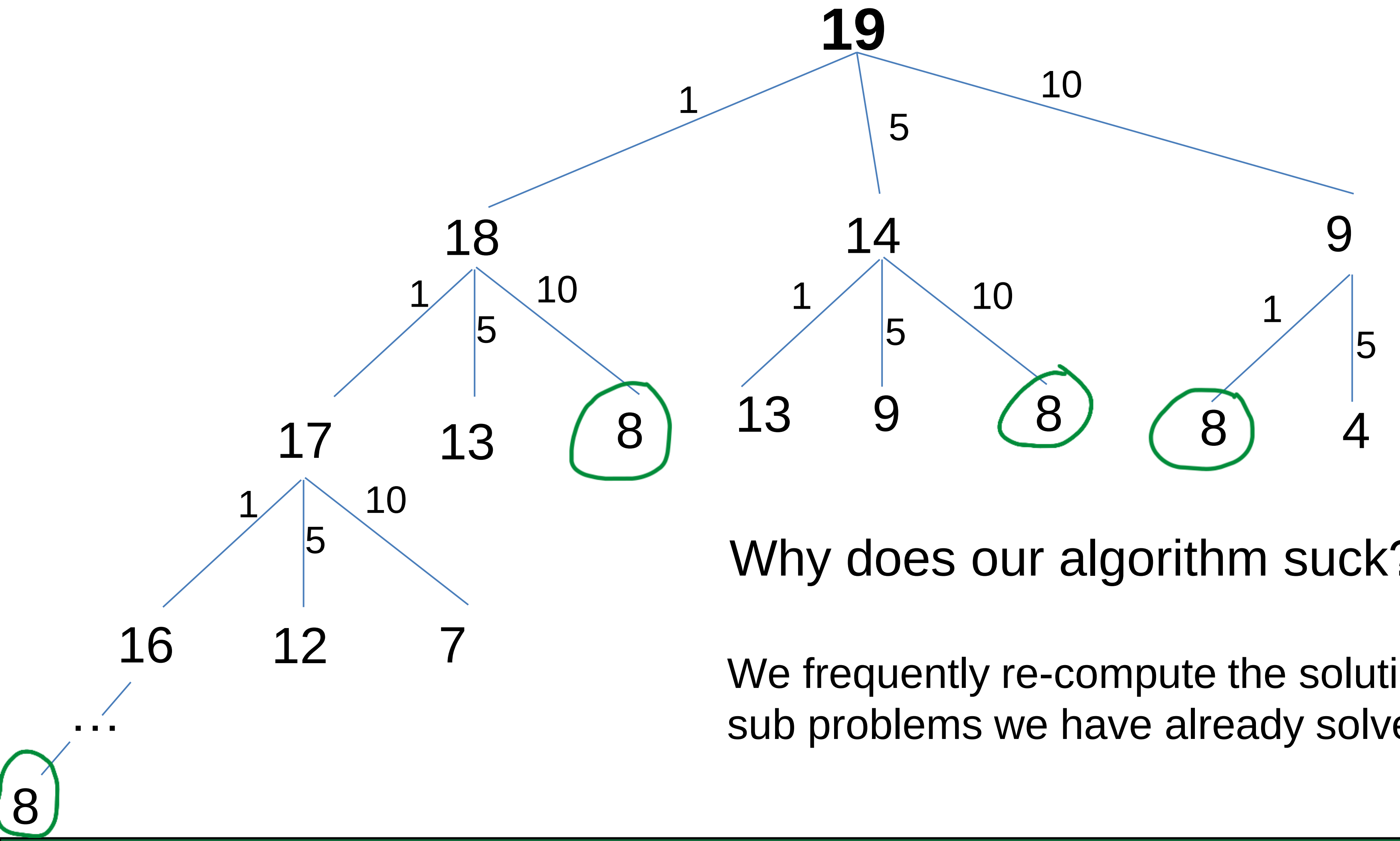
Change Making Problem



Why does our algorithm suck?

We frequently re-compute the solution to the sub problems we have already solved

Change Making Problem



We can fix this by utilizing some "smart recursion" AKA **Dynamic Programming**

Why does our algorithm suck?

We frequently re-compute the solution to the sub problems we have already solved

Dynamic Programming

Dynamic Programming is an algorithm technique used for optimization problems that involves smartly using recursion to solve a problem with many overlapping subproblems

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(If it has these two characteristics, we can use DP to solve it)

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1 **Optimal substructure-** an optimal solution can be constructed from optimal solutions of its sub problems

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(If it has these two characteristics, we can use DP to solve it)

① **Optimal substructure-** an optimal solution can be constructed from optimal solutions of its sub problems

② **Overlapping Subproblems-** we solve the same subproblem several times during the algorithm

① **Optimal substructure-** an optimal solution can be constructed from optimal solutions of its sub problems

The solution to the change making problem consists of solution to smaller change making problems ✓

② **Overlapping Subproblems-** we solve the same subproblem several times during the algorithm

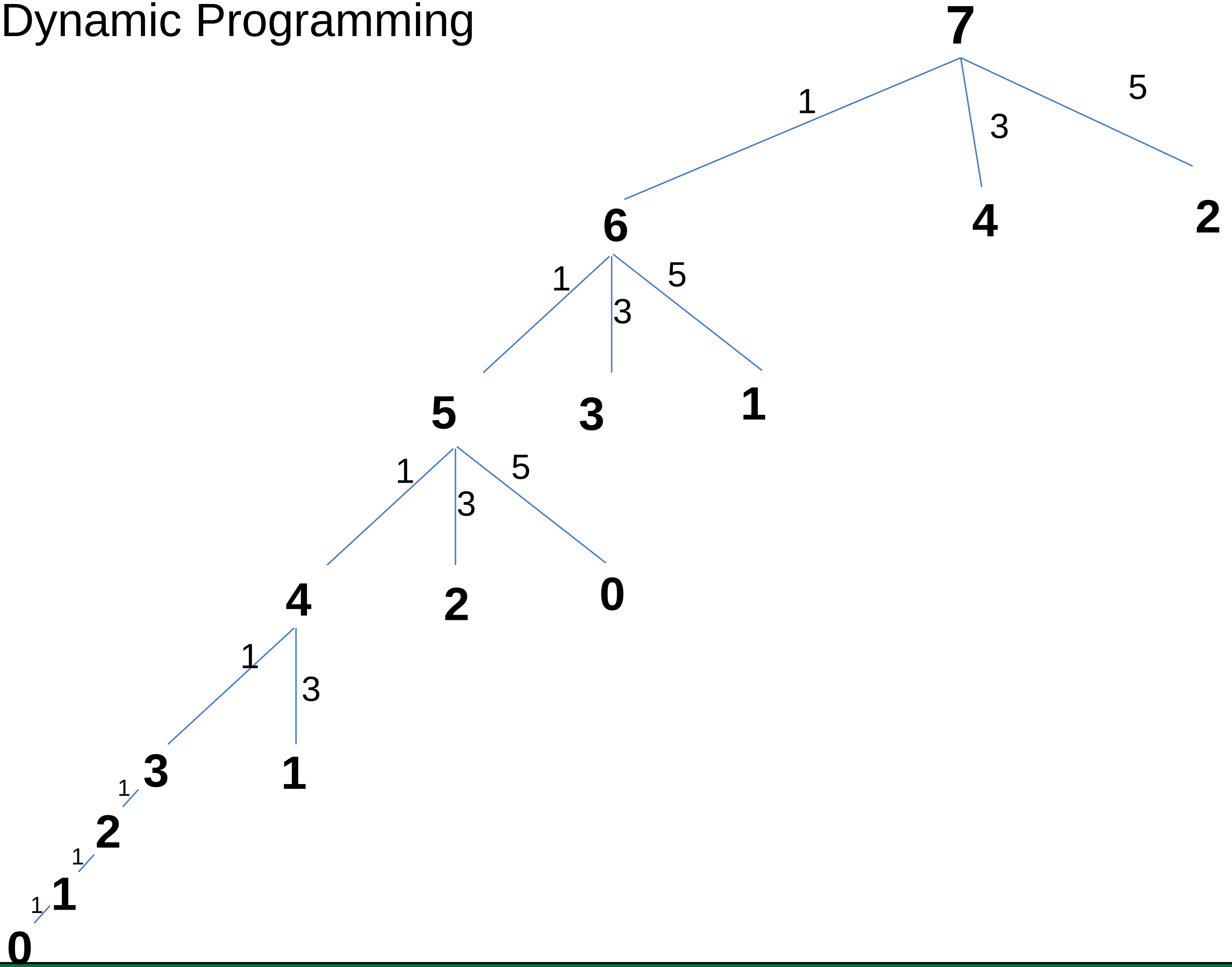
We frequently recompute the same subproblem throughout the algorithm ✓

We satisfy these two conditions, which means we can leverage **Dynamic Programming**

<u>Big idea of dynamic programming:</u>
Use **memoization** to store solutions of sub problems we have already solved, and don’t re compute them

(Yes, it’s “memoization” and not “memorization”)

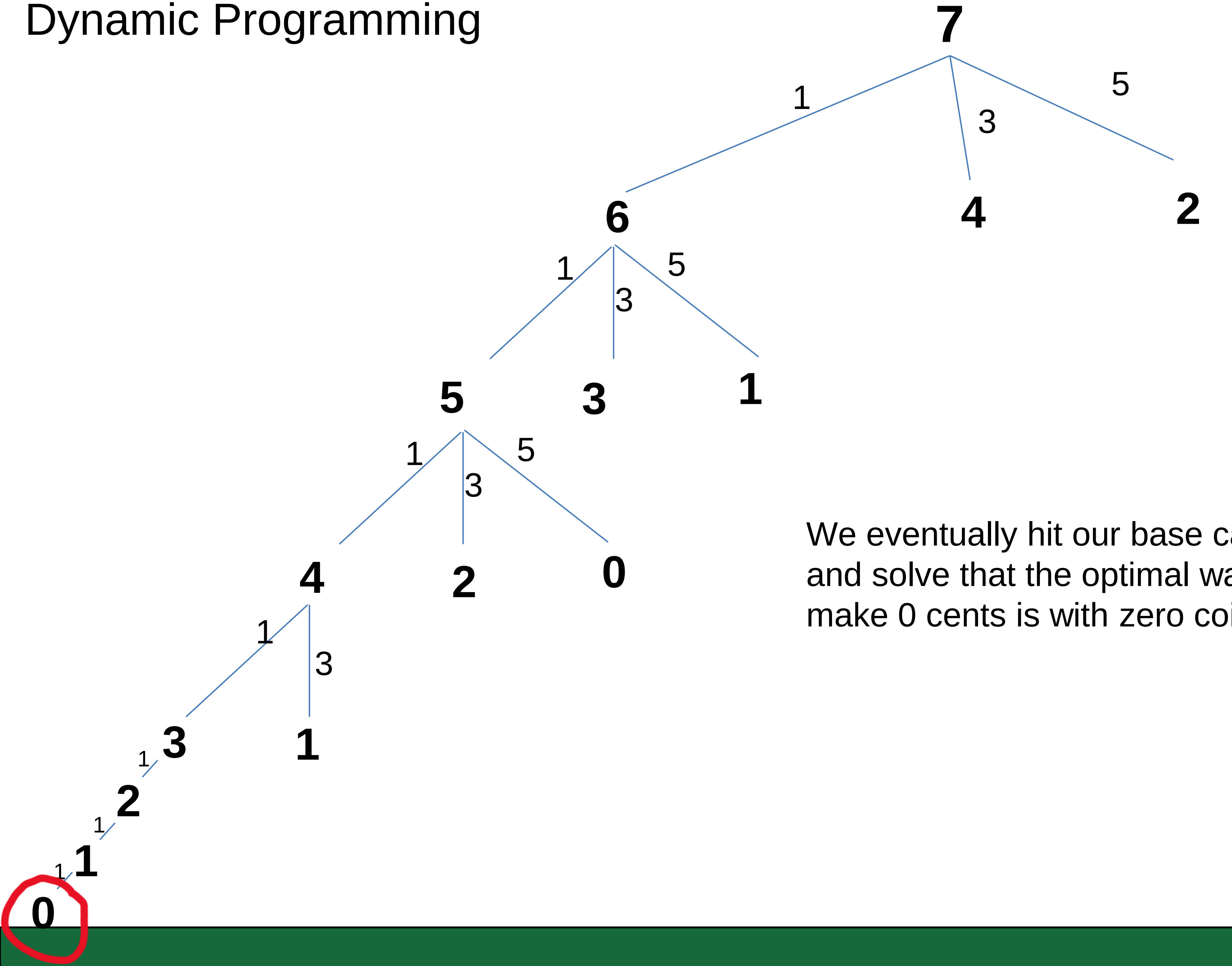
Dynamic Programming



Memoization Table

0	
1	
2	
3	
4	
5	
6	
7	

Dynamic Programming

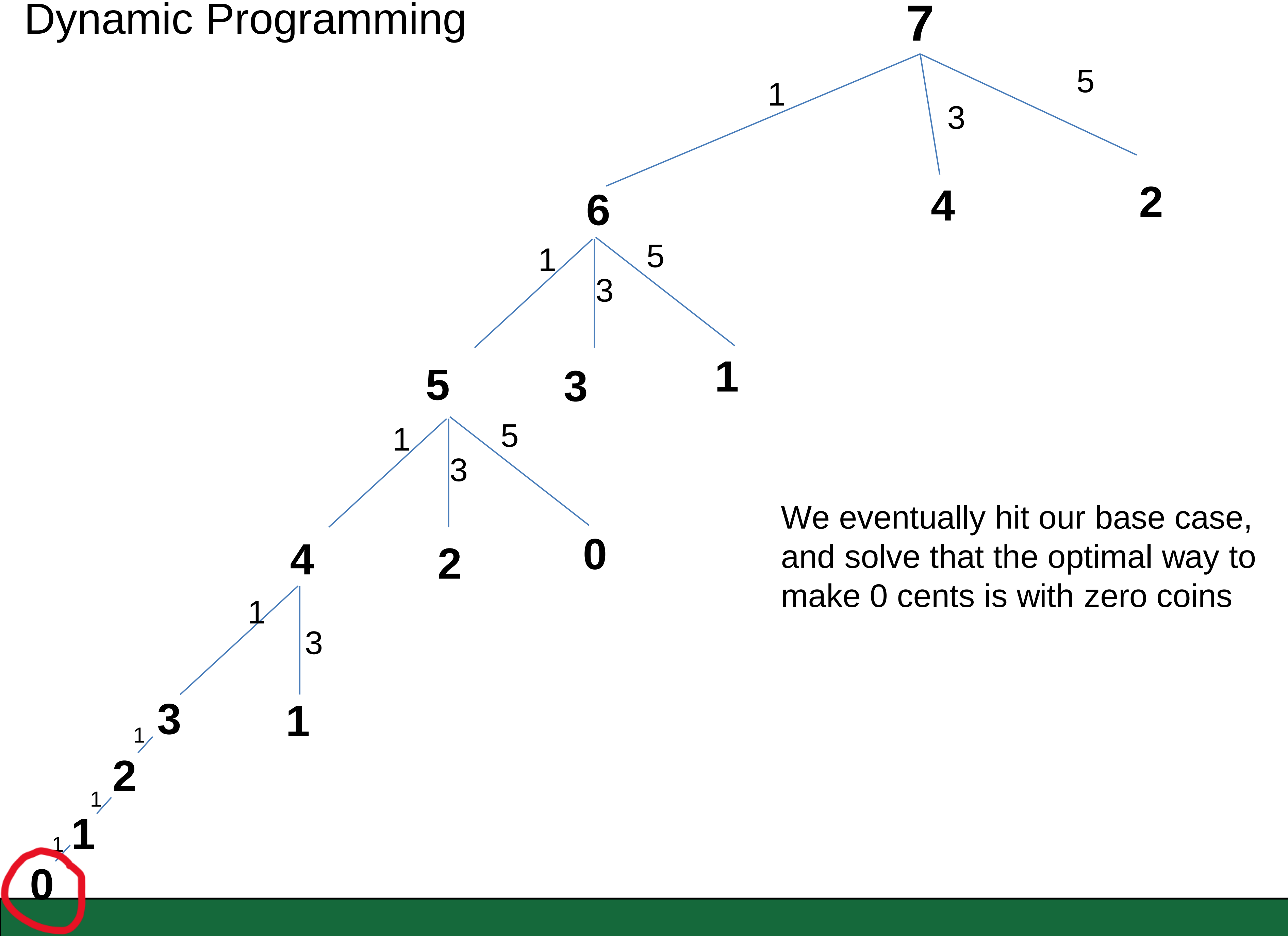


We eventually hit our base case, and solve that the optimal way to make 0 cents is with zero coins

Memoization Table

0	
1	
2	
3	
4	
5	
6	
7	

Dynamic Programming

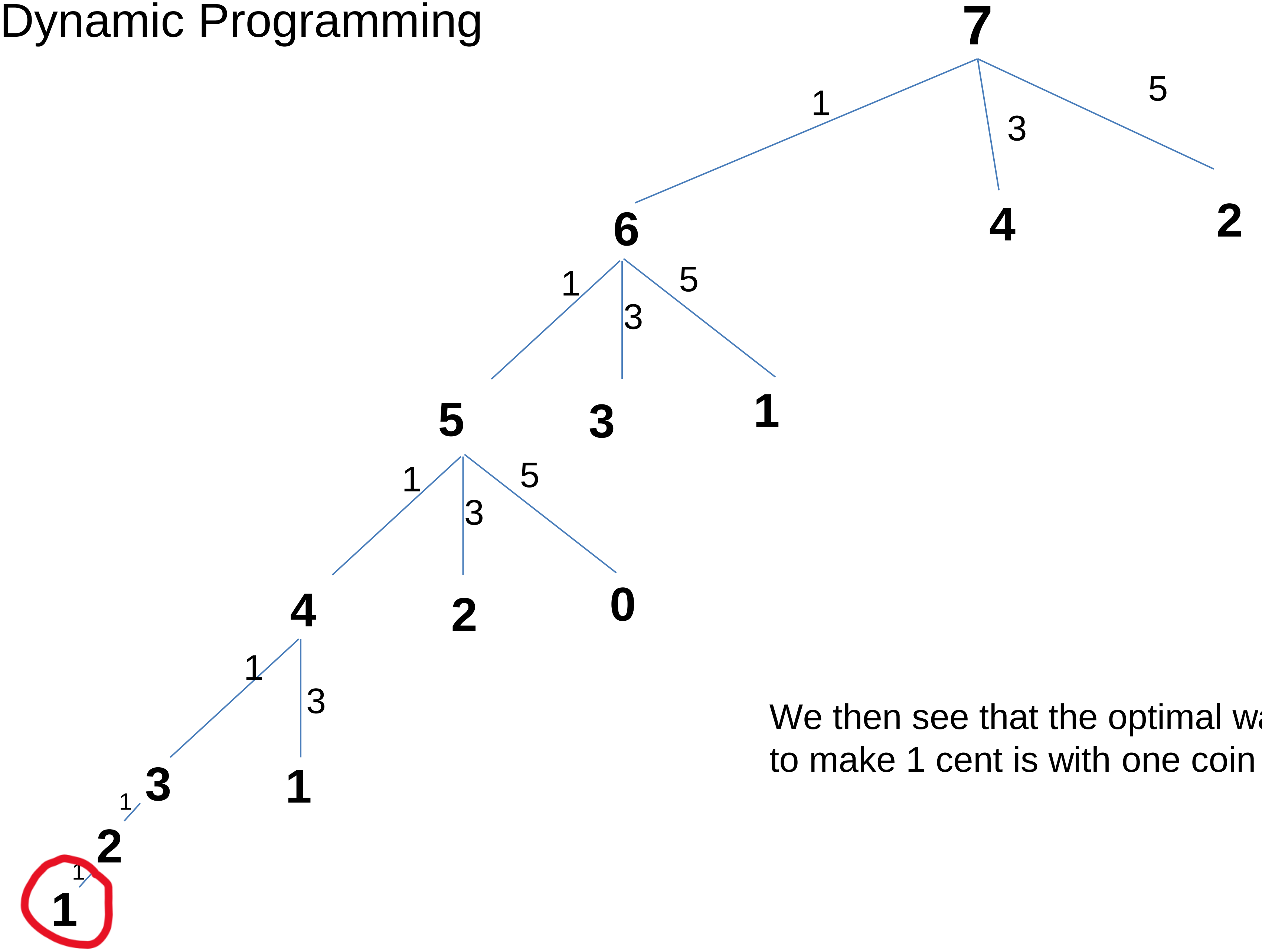


We eventually hit our base case, and solve that the optimal way to make 0 cents is with zero coins

Memoization Table

0	0
1	
2	
3	
4	
5	
6	
7	

Dynamic Programming



Memoization Table

0	0
1	1
2	
3	
4	
5	
6	
7	

We then see that the optimal way to make 1 cent is with one coin

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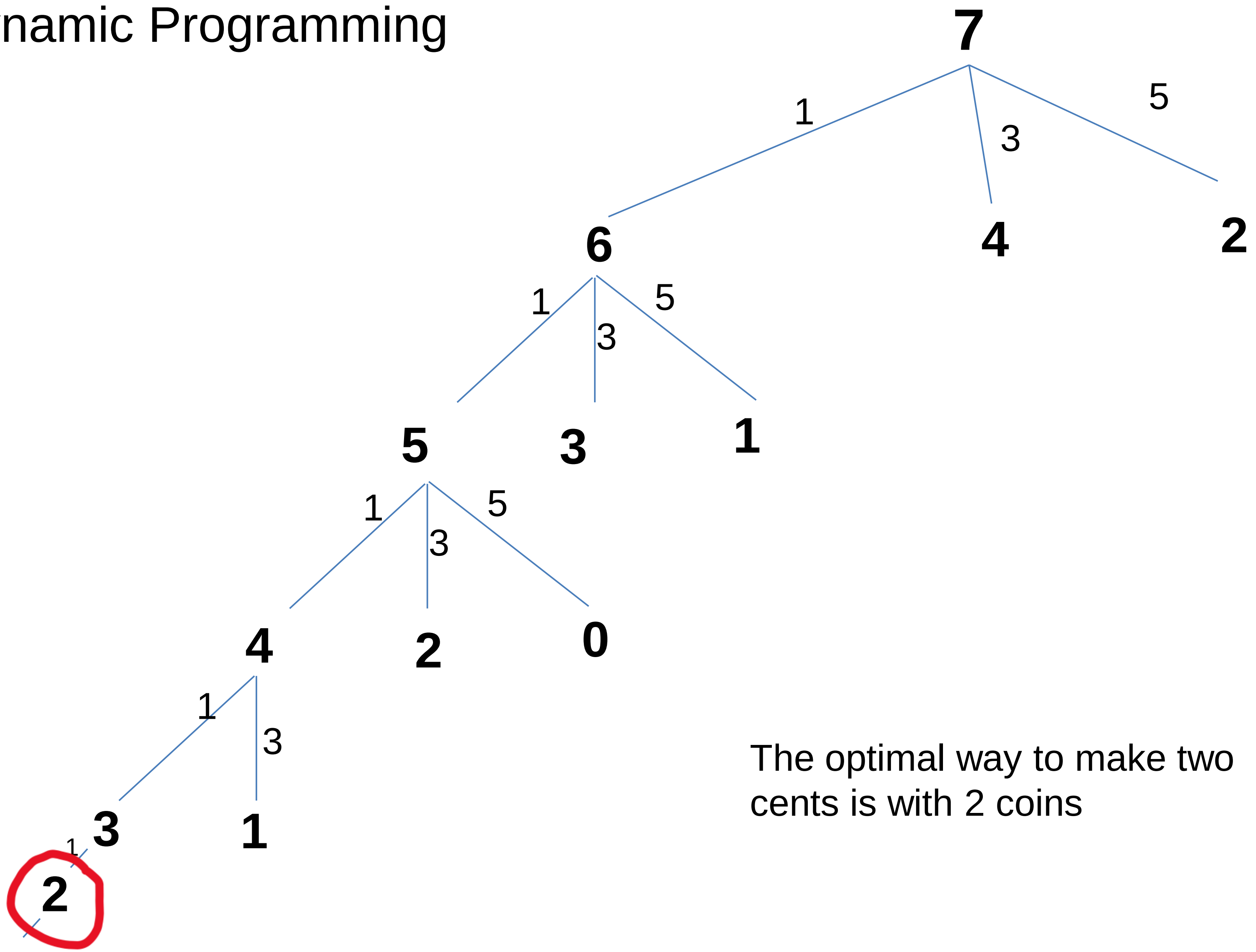
Dynamic Programming

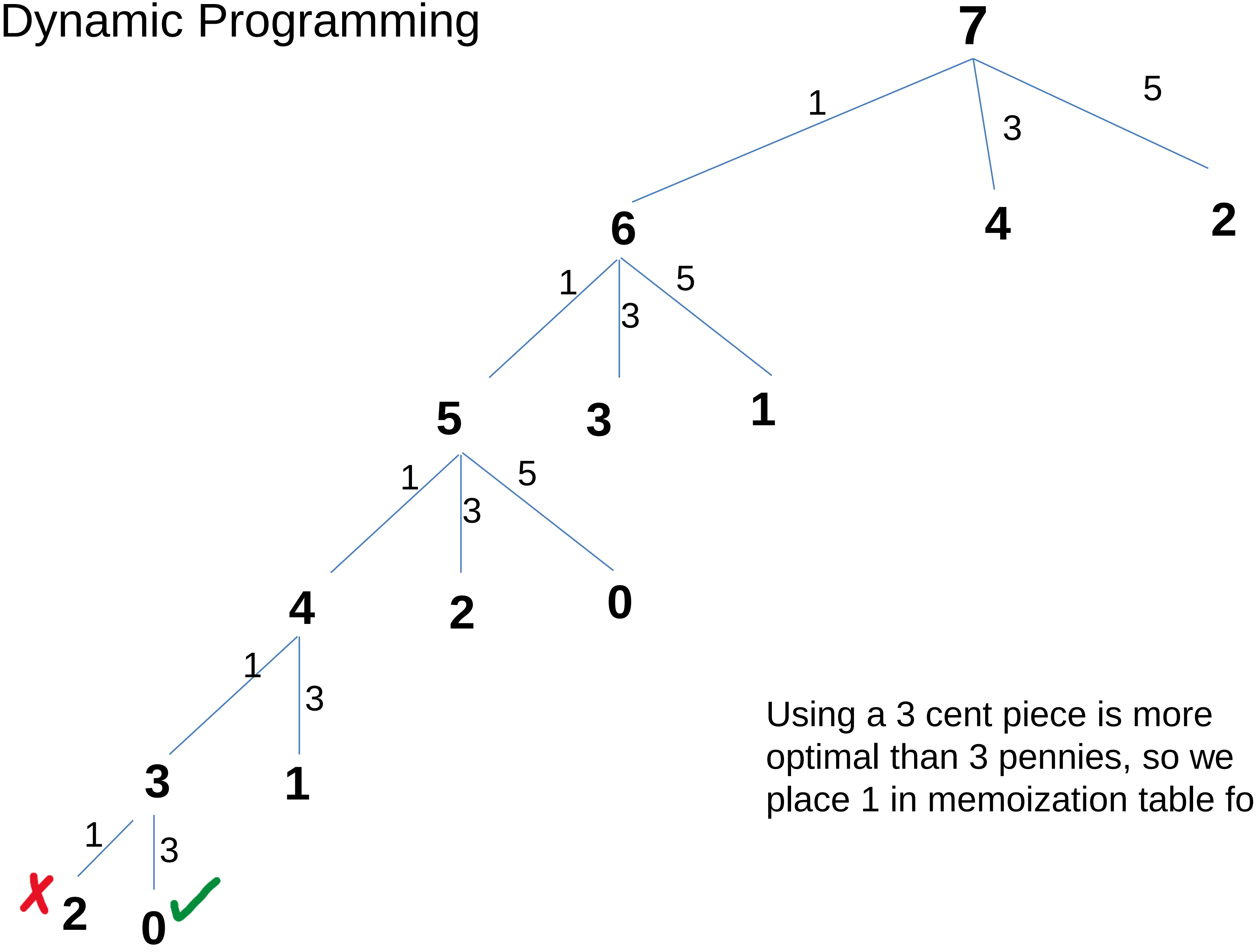


Memoization Table

0	0
1	1
2	2
3	
4	
5	
6	
7	

The optimal way to make two cents is with 2 coins

Dynamic Programming

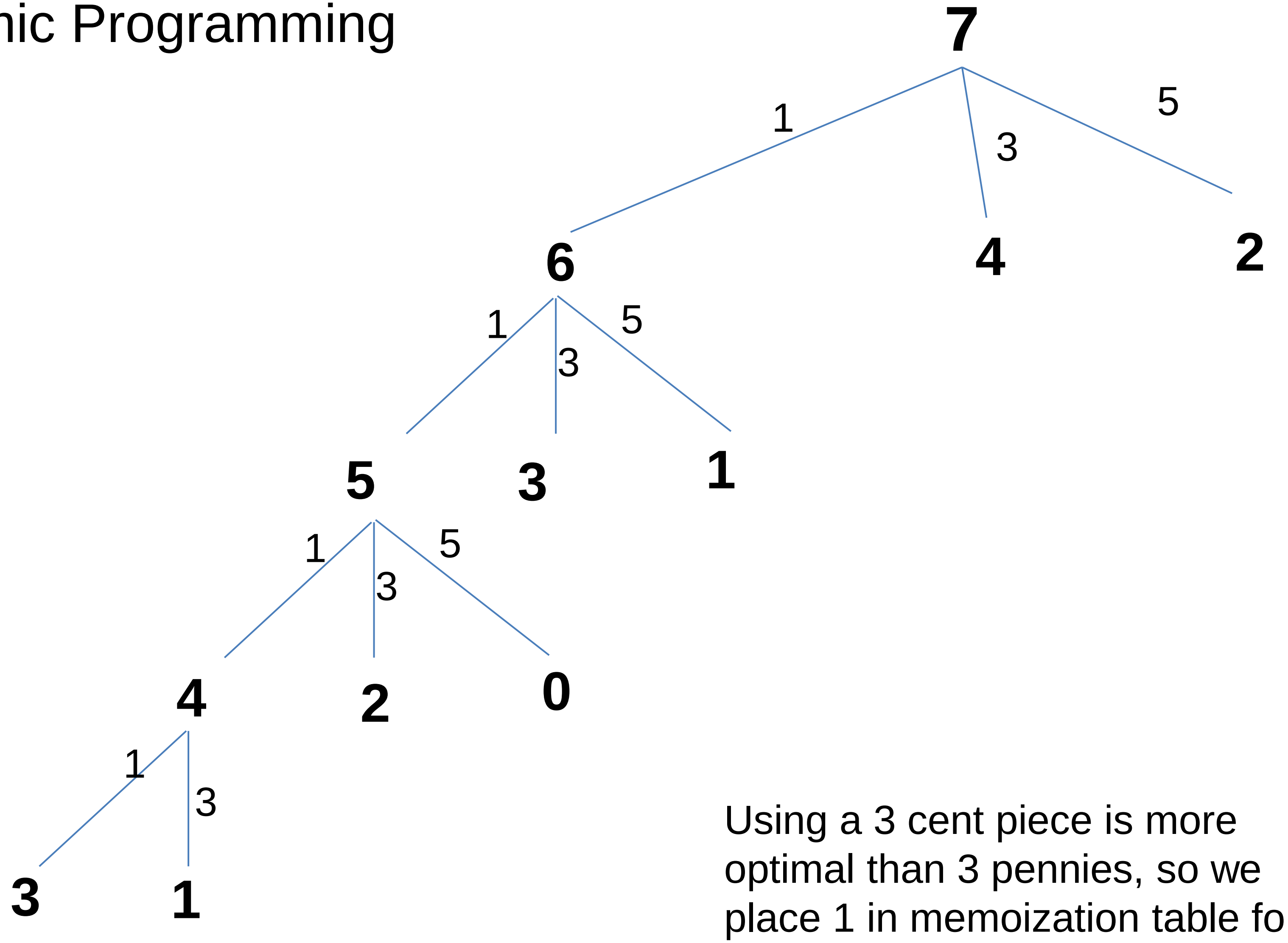


Memoization Table

0	0
1	1
2	2
3	
4	
5	
6	
7	

Using a 3 cent piece is more optimal than 3 pennies, so we place 1 in memoization table for 3

Dynamic Programming

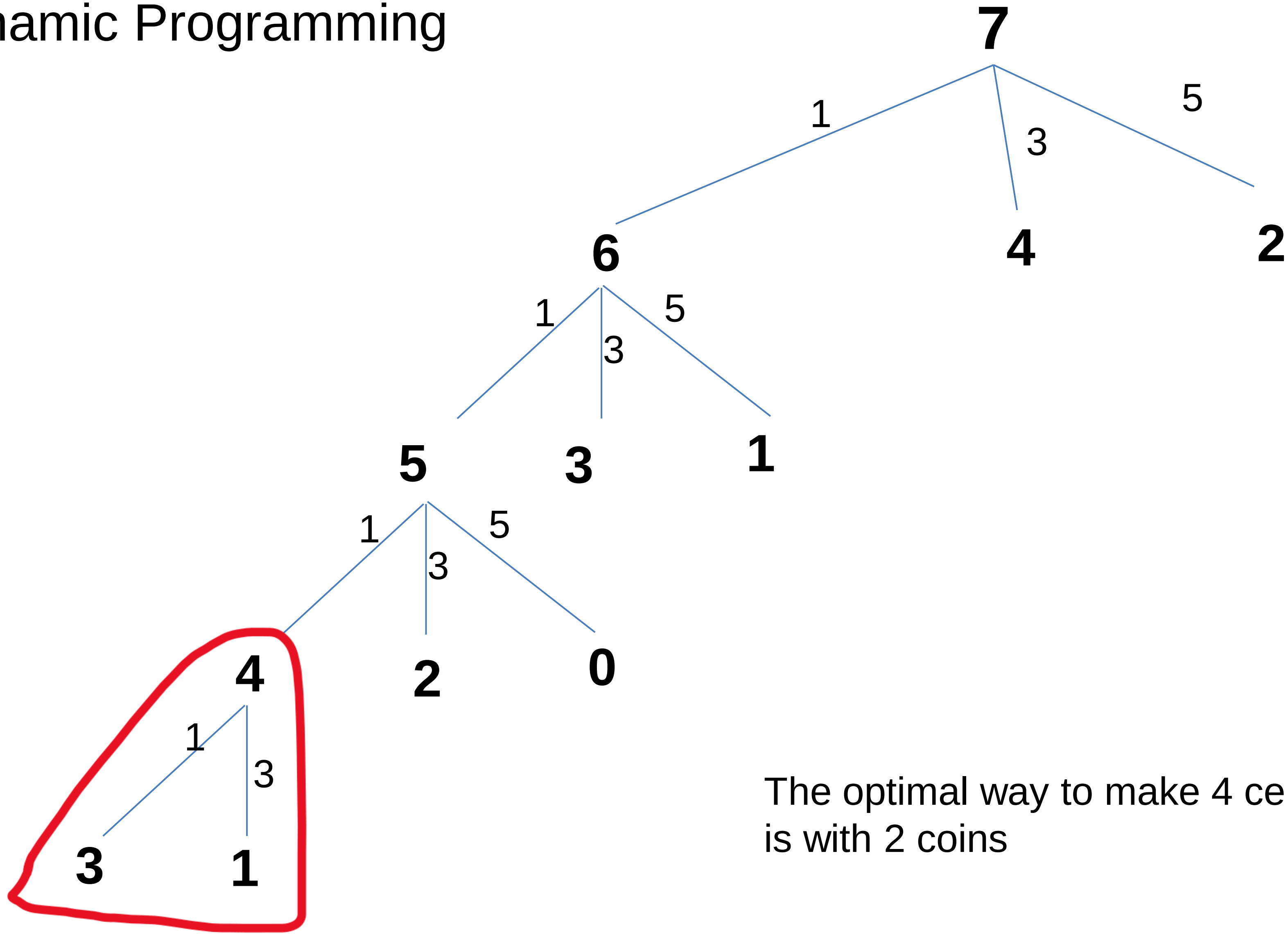


Memoization Table

0	0
1	1
2	2
3	1
4	
5	
6	
7	

Using a 3 cent piece is more optimal than 3 pennies, so we place 1 in memoization table for 3

Dynamic Programming

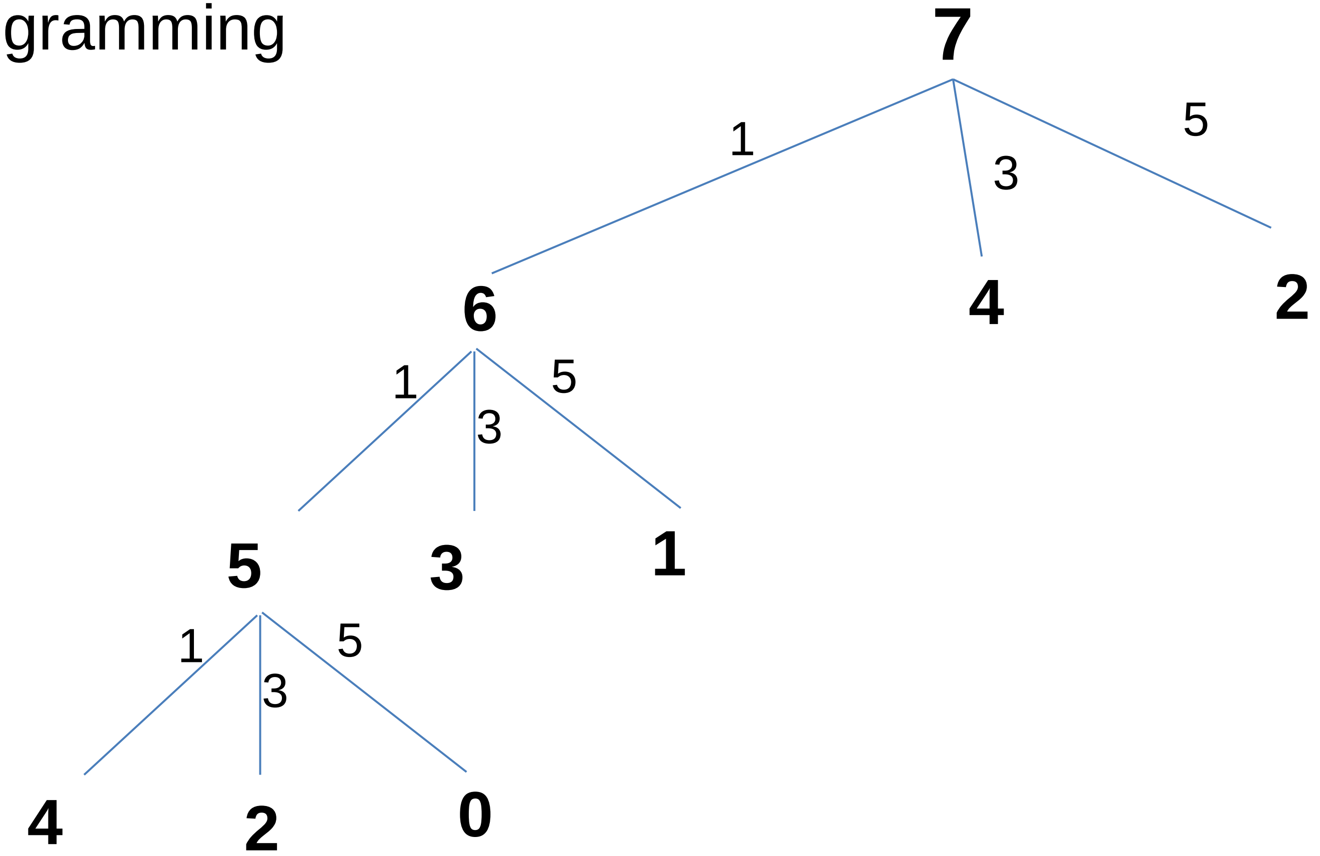


Memoization Table

0	0
1	1
2	2
3	1
4	
5	
6	
7	

The optimal way to make 4 cents
is with 2 coins

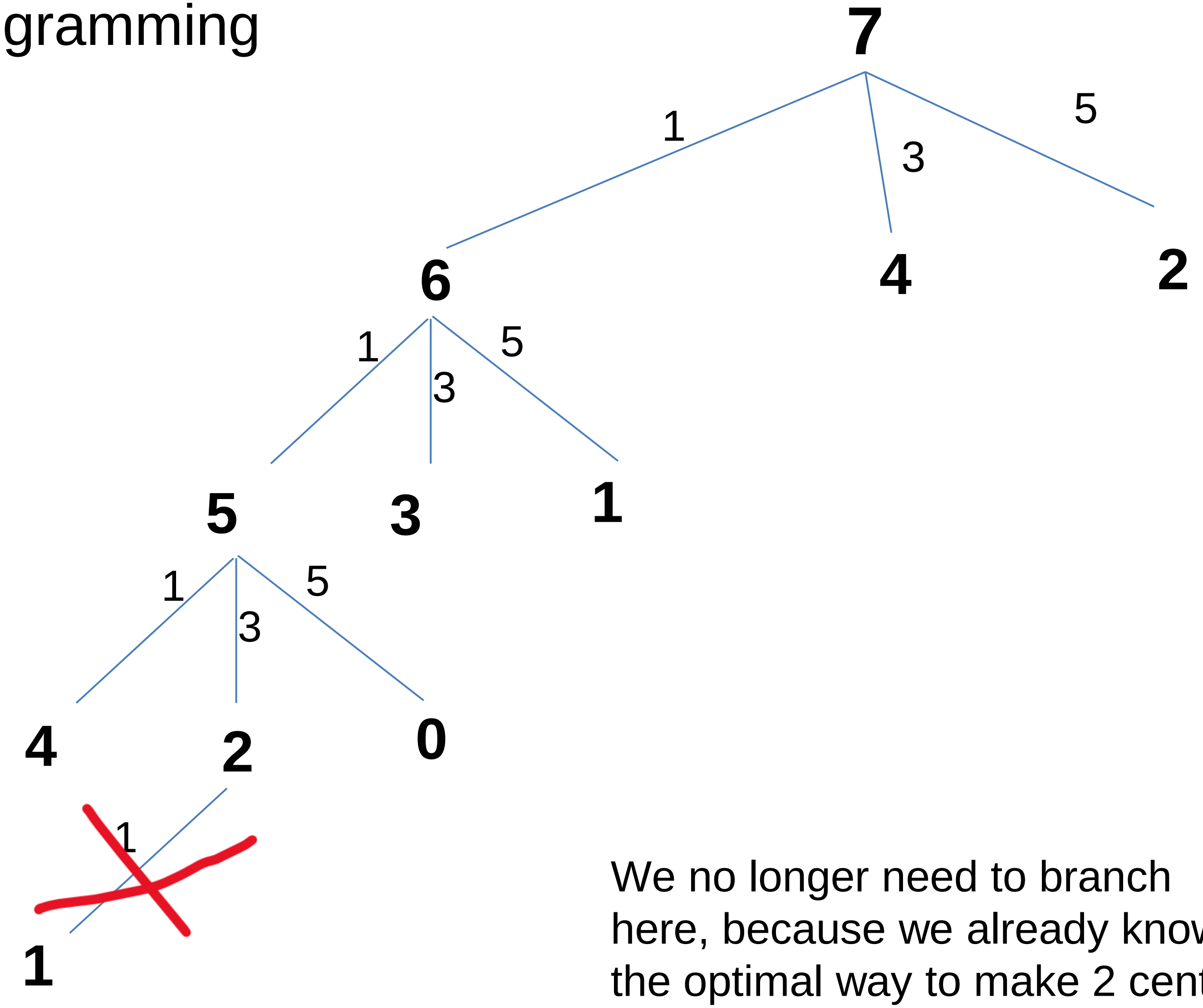
Dynamic Programming



Memoization Table

0	0
1	1
2	2
3	1
4	2
5	
6	
7	

Dynamic Programming

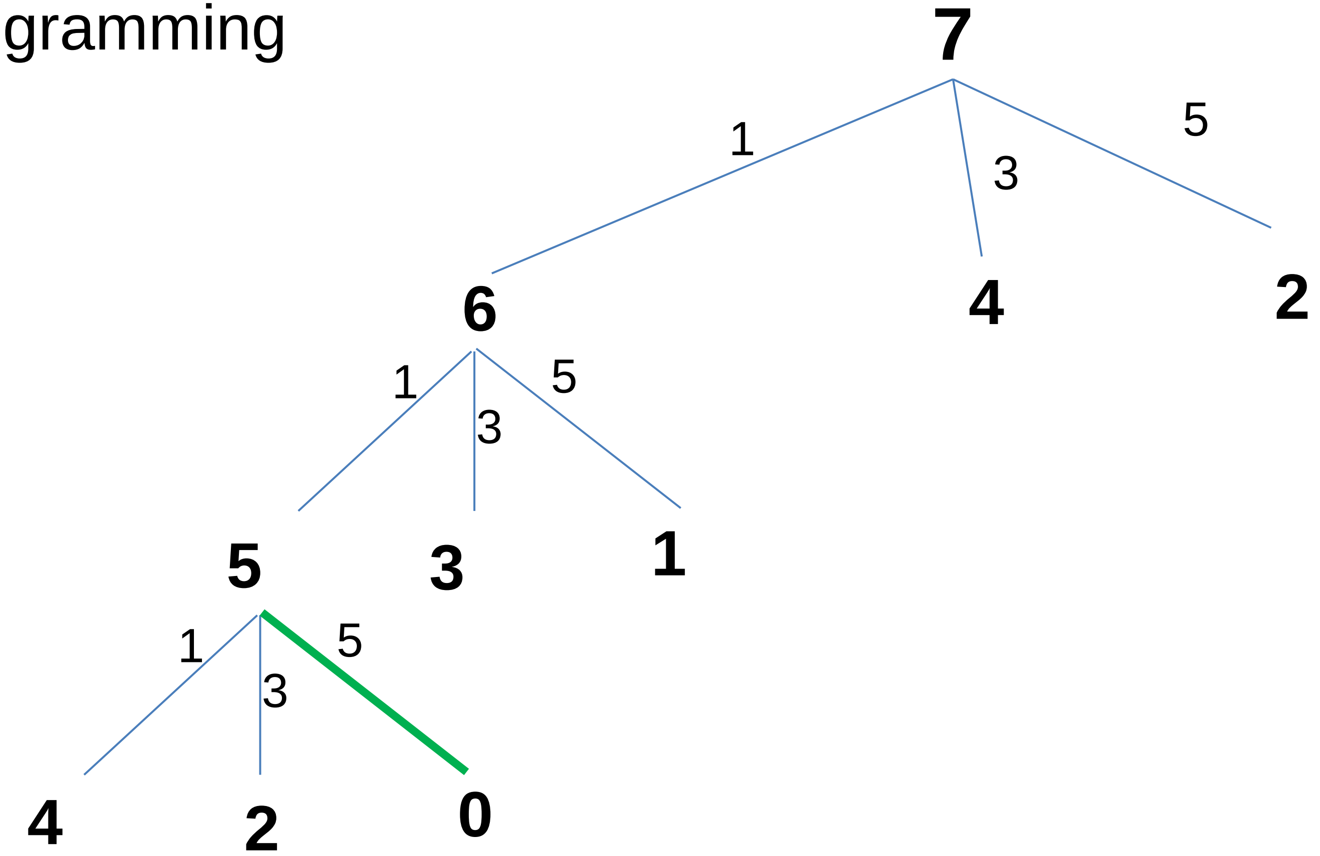


Memoization Table

0	0
1	1
2	2
3	1
4	2
5	
6	
7	

We no longer need to branch here, because we already know the optimal way to make 2 cents!

Dynamic Programming

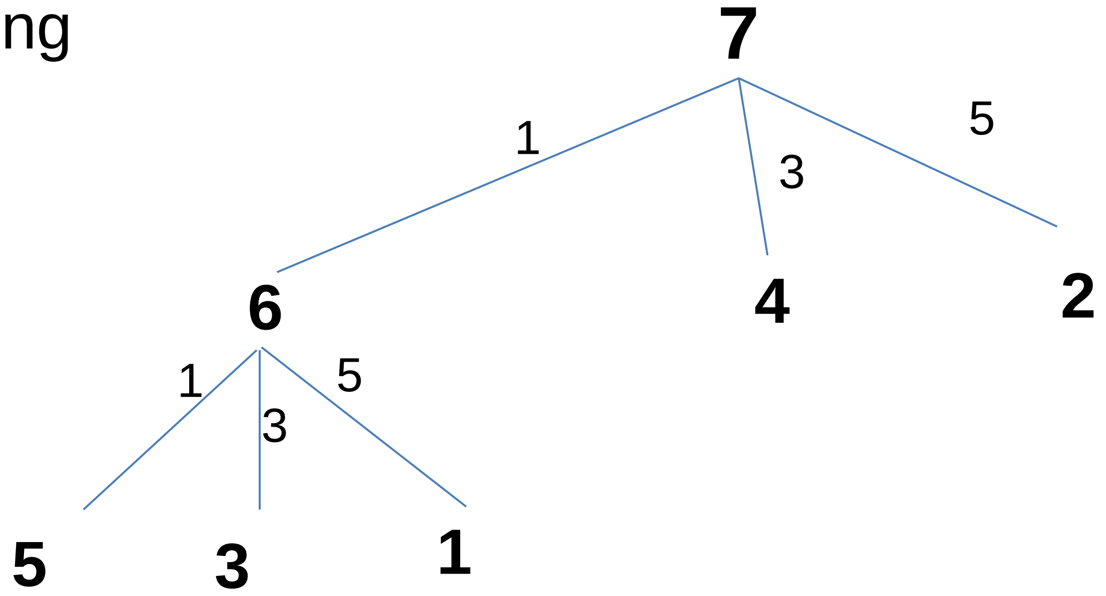


Memoization Table

0	0
1	1
2	2
3	1
4	2
5	
6	
7	

We learn the optimal way to make five cents is with one coin

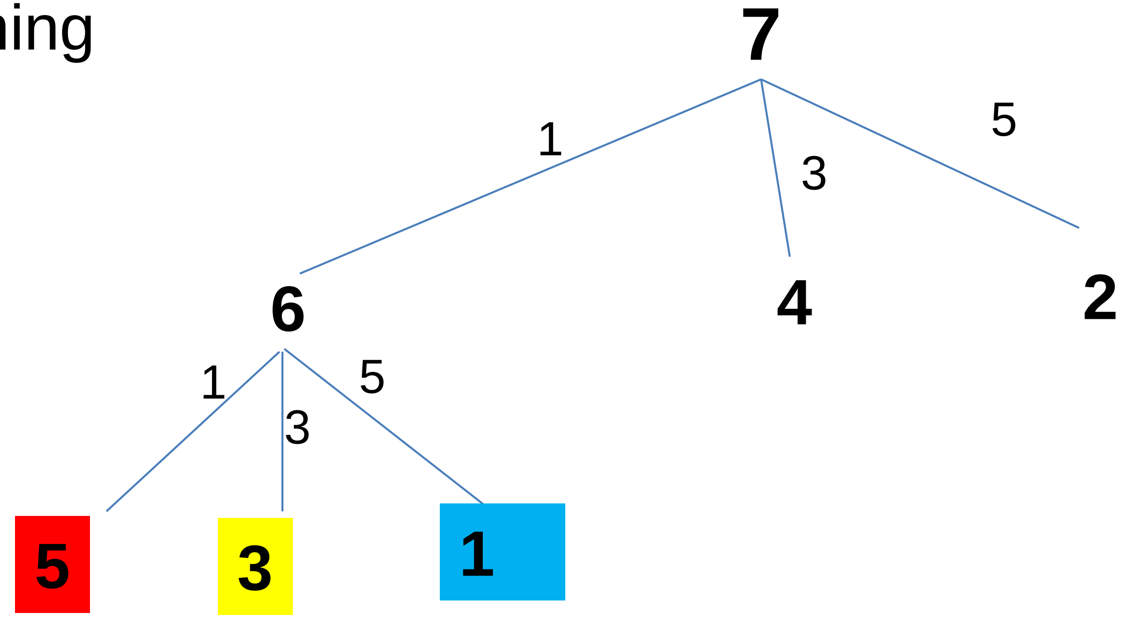
Dynamic Programming



Memoization Table

0	0
1	1
2	2
3	1
4	2
5	1
6	
7	

Dynamic Programming

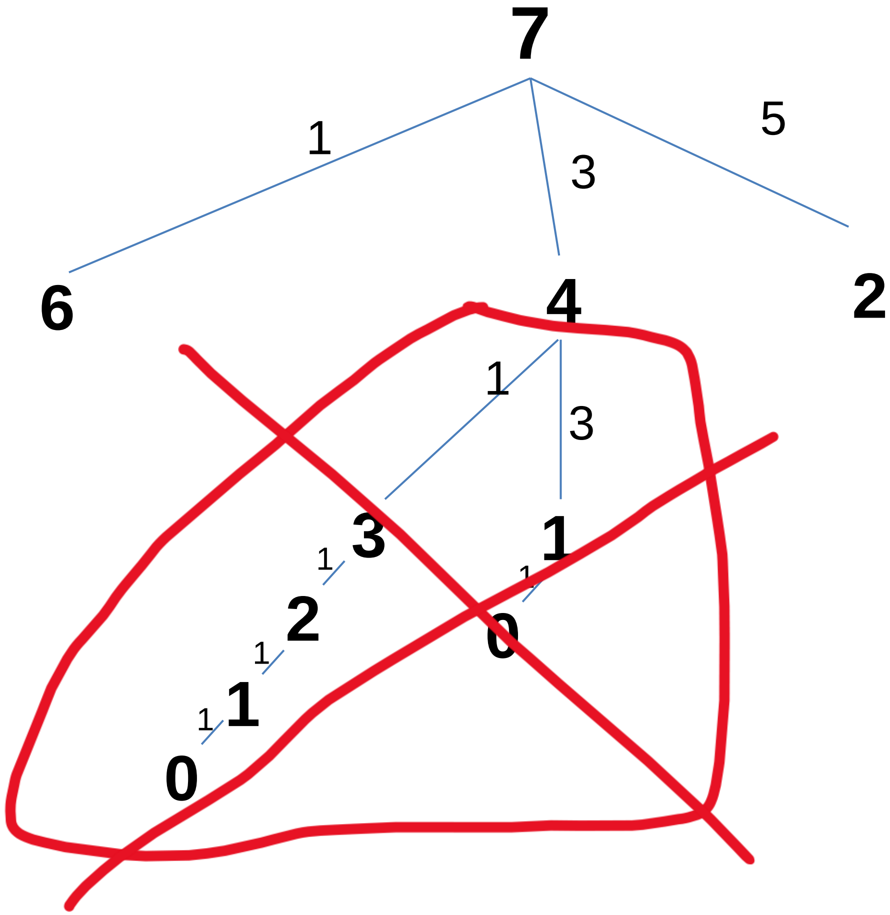


All three of these branches have the same cost, so making 6 cents requires 2 coint

Memoization Table

0	0
1	1
2	2
3	1
4	2
5	1
6	
7	

Dynamic Programming

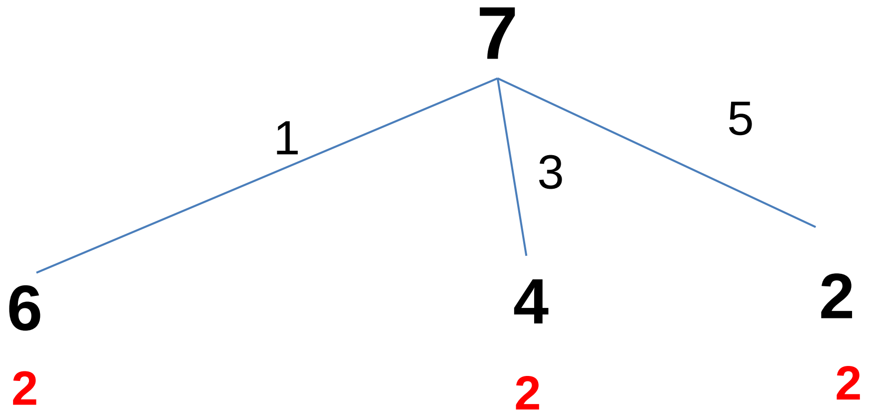


Memoization Table

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	

We no longer need to branch here, because we already know the optimal solution for 4, so just check our memoization table!

Dynamic Programming



Memoization Table

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	

7

Memoization Table

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	3

The optimal way to make 7 cents is with 3 coins:

[1, 1, 5]

[3, 1, 3]

Code

7

Memoization Table

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	3

Top – Bottom Dynamic Programming (Memoization)

(Use Recursion)

Dynamic Programming (Bottom to Top)

If we don't want to use recursion, we can use a for loop to fill out this entire array (**tabulation**)

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	3

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

0	0
1	∞
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

0	0
1	1
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

0	0
1	1
2	2
3	∞
4	∞
5	∞
6	∞
7	∞

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

0	0
1	1
2	2
3	1
4	∞
5	∞
6	∞
7	∞

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

	Cache
0	0
1	1
2	2
3	1
4	2
5	∞
6	∞
7	∞

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3, 5)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

0	0
1	1
2	2
3	1
4	2
5	∞
6	∞
7	∞

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3, 5)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

0	0
1	1
2	2
3	1
4	2
5	1
6	∞
7	∞

Dynamic Programming (Bottom to Top)

cache[0] = 0 //base case
For each cent value, **i**, in our array (0 – P):
    For each coin in our denomination set **c**: **(1, 3, 5)**
        if(c – cache[i] >= 0):
            x = cache[i – c];
            if **x** is smaller than what is currently in the cache:
                update cache to be **x + 1**

Cache

	Cache
0	0
1	1
2	2
3	1
4	2
5	1
6	∞
7	∞

Dynamic Programming (Bottom to Top)

Cache

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	∞

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3, 5)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3, 5)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	∞

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3, 5)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	3

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3, 5)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1
```

Cache

	Cache
0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	3

Our table is filled out, we can now query it!

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3, 5)
                if(c – cache[i] >= 0):
                        x = cache[i – c];
                        if x is smaller than what is currently in the cache:
                                update cache to be x + 1

return cache[P]
```

Cache

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	3

Our table is filled out, we can now query it!

Dynamic Programming (Bottom to Top)

```
cache[0] = 0 //base case
For each cent value, i, in our array (0 – P):
        For each coin in our denomination set c: (1, 3, 5)
                if(c – cache[i] >= 0):
                        x = cache[i – c]
                        if x is sm... ...hat is currently in the cache:
                        ...e cache to be x + 1

return ca...
```

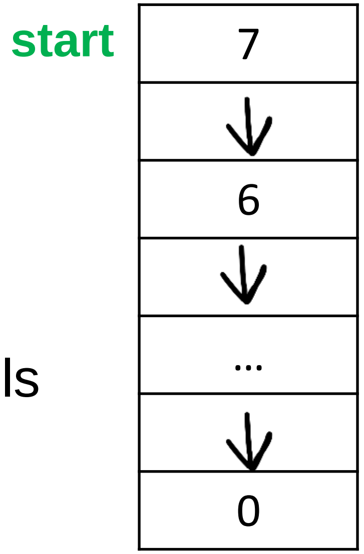
Bottom-Up Dynamic Programming (Tabulation)

Our table is filled out, we can now query it!

Cache

0	0
1	1
2	2
3	1
4	2
5	1
6	2
7	3

Dynamic Programming

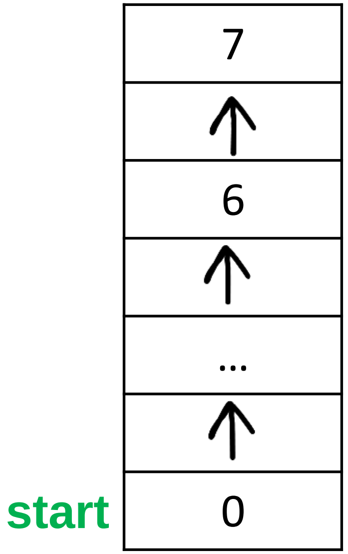


Top to Bottom Dynamic Programming

→ Use recursion, and fill out a **memoization table** as you are making recursive calls

Bottom-Up Dynamic Programming

→ Use a for loop to fill out a table (**tabulation**), then query the table

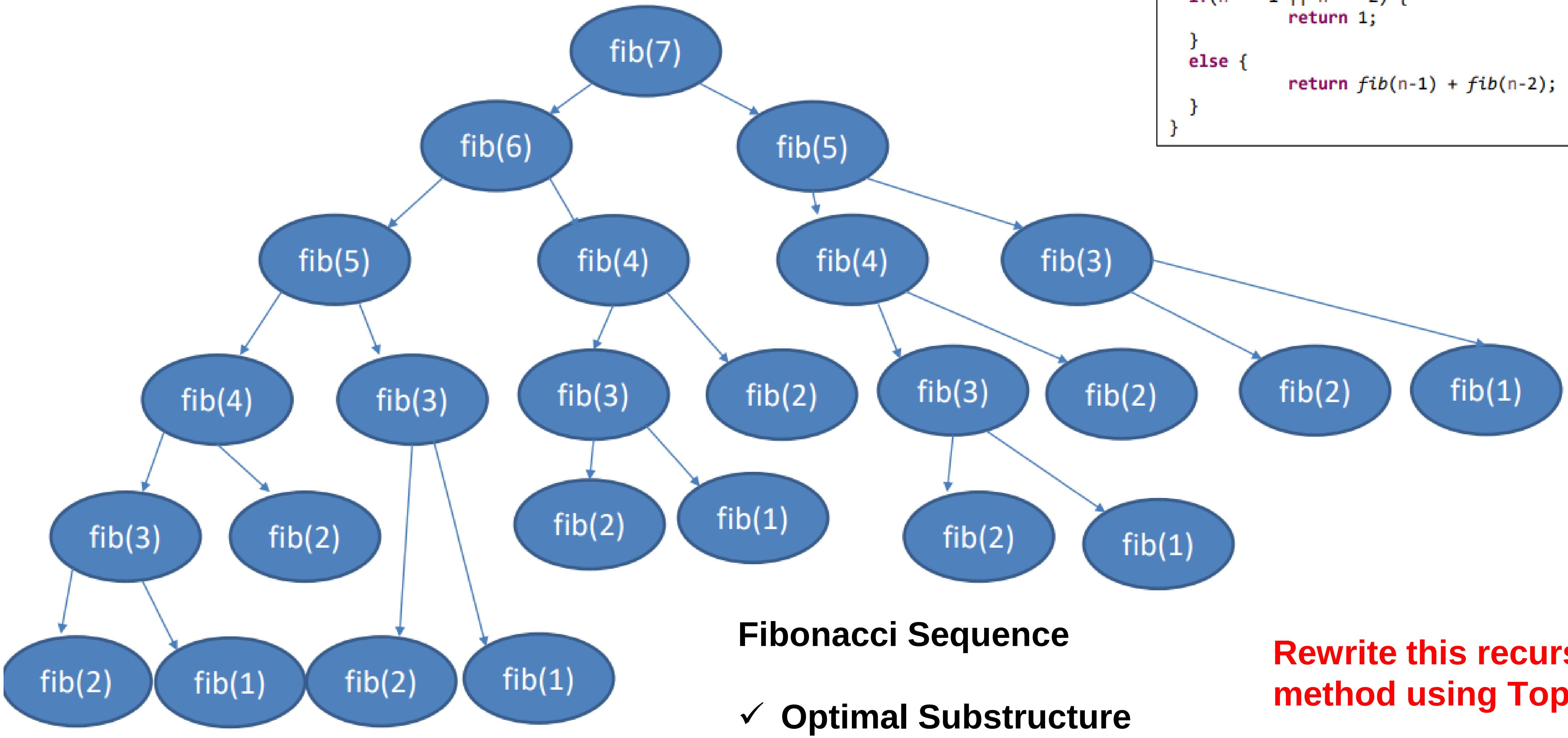


Both have the same running time. But a computer can handle a for loop better than recursion

(I think recursion is easier to understand)

DP improves running time from exponential to **O(len(D) * p)**

Number of recursive calls = 24




```java
private static int fib(int n) {
  if(n == 1 || n == 2) {
            return 1;
  }
  else {
            return fib(n-1) + fib(n-2);
  }
}
```

Fibonacci Sequence

- ✓ **Optimal Substructure**
- ✓ **Overlapping Subproblems**

Rewrite this recursive method using Top-Down DP