Definitions: Group Exercises

CSCI 246

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Problem 1. Consider the following definitions of *even*:
Definition 1. An integer x is even if and only if 2 *divides* x.
Definition 2. An integer x is even if and only if

1. $x = 0$,
2. $0 < x$ and $x - 2$ is even, or
3. $0 > x$ and $x + 2$ is even

A. why might you prefer one definition over the other?
Ans. Definition 1 is simple ane makes re-use of the existing definition of *devides.* Definition 2, on the other hand, uses a recursive definition that may be more useful for proving facts about even numbers recursively.

B. Are both definitons well defined? Why or why not?
Ans. No, due to the recursive structure of Definition 2, it is not clear weather any *odd* (in the typical mathematical defintion $\{\dots, -3, -1, 1, 3, \dots\}$) satisfies Definition 2. For example, -1 is *even* if and only if 1 is *even* and 1 is *even* if and only if -1 is even, ... Definition 2, doesn't tell us if we should assume infinite derivations of *even* is or is not consider valid.

Bonus. [Advanced Concept]. One can make Definition 2 well formed by defining if infinite derivations are valid. We can modify Definition 2 to explicitly reject infinite derivations of the *evenness* of an integer, by saying that *even* is the *least solution* of rules 1, 2, and 3. Then, Definition 2 is well formed and equivalent to Definition 1, in that both definitions of *even* agree on which integers are *even.*

Problem 2. Provide three definitions of *odd*:
A. Give a definition of *odd* in terms of *even*.
Answers may vary:
Definition 1. An integer x is *odd* if and only if x is not *even.*
Definition 2. An integer x is *odd* if and only if $x + 1$ is *even.*

B. Give a definition of *odd* using divisibility.
Answers may vary:
Definition 1. An integer x is *odd* if and only if x is not *divisible* by 2.
Definition 2. An integer x is *odd* if and only if $x + 1$ is *divisible* by 2.

C. Give a definition of *odd* using just multiplication and addition.
Definition An integer x is *odd* if and only if there is an integer c such that $x = 2c + 1$.

D. Using each definition, try to show that 1, -1, 3, and 101 are *odd*. Can you show that 2 is not *odd*?
Solutions for Definitions A.2, B.2, and C are all similar. I.e.,

$$1 = 2 \times 0 + 1 \qquad -1 = 2 \times -1 + 1 \qquad 3 = 2 \times 1 + 1 \qquad 101 = 2 \times 50 + 1$$

Definitions A.1 and B.1 can not be shown as both requires showing a number is not even, which would require showing e.g., for 1 there is no integer c such that $1 = 2 \times C$.

On the other hand, A.1 and B.1 can be used to easily show that 2 is not odd by showing 2 is *even.*

Problem 3. Consider the following two definitions of divides:

Definition 1. For integers a and b, a divides b if and only if there is an integer c such $b = ac$.

Definition 2. For integers a and b, a divides b if and only if $\frac{b}{a}$ is an integer.

Are these definitions equivalent? I.e., for all integers a and b the two definitions agree? If not, give and a and b such that definition 1 and 2 disagree.

Ans. No, Definition 1 states that 0 *devides* 0, while Definition 2 does not. I.e., $0 = 0c$ for any integer c, but $\frac{0}{0}$ is undefined.

Problem 4. Consider the following definition of *prime*.

Definition. We say p is *prime* if and only if p is a positive integer greater than 1 (i.e., $1 < p \in \mathbb{Z}$) and the only positive divisors of p are 1 and p itself.

Show that the following values are not *prime*:

A. 21 No, 21 is *divisible* by both 3 and 7.
A. 0 No, $0 \leq 1$ and cannot be prime by definition.
A. π No, π is not an integer and cannot be prime by definition.
A. $\frac{1}{2}$ No, $\frac{1}{2}$ is not an integer and cannot be prime by definition.
A. -2 No, $-2 \leq 1$ and cannot be prime by definition.
A. 1 No, $1 \leq 1$ and cannot be prime by definition.

Problem 5. Consider the following definition of *composite*.

Definition. We say that c is *composite* if and only if c is a positive integer greater than 1 (i.e., $1 < p \in \mathbb{Z}$) and there is some positive integer b greater than 1 and less than c that divides c (i.e., $1 < b < c$ and b divides c.) .

Give five integers that are both not *prime* and not *composite*.
Ans. Give any 5 integers less than or equal to 1; e.g., $-3, -2, -1, 0, 1$.

Problem 6. Give a definition for when a is the (integer) square root of b—i.e., 0 is the square root of 0, 1 is the square root of 1, 2 is the square root of 4, ...

Definition 1.
Ans. The integer x is the integer square root (denoted *isqrt*) of the integer y if and only if $y = x^2$.

Give another definition, which additionally defines that 1 is the square root of 3, 2 is the square root of 5, 6, and 7, ...

Definition 2.
Ans. The integer x is the integer square root of the integer y if and only if $x^2 \leq y < (x+1)^2$.

Explanation. In effect, the integer square root of y is the square root of the largest perfect square less than or equal to y. We encode this into our definition by requiring that y is sandwiched between x^2 and $(x+1)^2$, both of which are perfect squares.

For which pairs of integers to definitions 1 and 2 agree?
Ans. Definition 1 and 2 agree on all integers x and y such that $y = x^2$ (i.e., when y is a perfect square).