Proofs: Group Exercises

CSCI 246

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Problem 1. Prove that the sum of two *odd* numbers is *even*.

Answers may vary: I provide two possible solutions. One using a structured, step-by-step derivation, and another streamlined proof written in prose.

Proposition 1. If $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ are *odd*, then $a + b$ is *odd*.

Proof.

1. Let a and b be *odd* integers. (by hypothesis)
2. There is some $k_a \in \mathbb{Z}$ such that $a = 2k_a + 1$. (by 1 & definition of *odd*)
3. There is some $k_b \in \mathbb{Z}$ such that $b = 2k_b + 1$. (by 1 & definition of *odd*)
4. $a + b = a + b$. (by reflexivity of $=$)
5. $a + b = (2k_a + 1) + b$. (by 2 & 4)
6. $a + b = (2k_a + 1) + (2k_b + 1)$. (by 3 & 5)
7. $a + b = 2(k_a + k_b + 1)$. (by 6 & simplification)
8. $k_a + k_b + 1$ is an integer. (sum of integers is an integer)
9. $2|(a + b)$. (by 7, 8, & definition of *divides*)
10. $a + b$ is *even* (by 9 & definition of *even*)

□

Proof. Let a be an *even* integer and b an *odd* integer. By definition there must be some $k_a \in \mathbb{Z}$ and $k_b \in \mathbb{Z}$ such that $a = 2k_a + 1$ and $b = 2k_b + 1$. Necessarily, $a + b = (2k_a + 1) + (2k_b + 1)$. By simplification, we have $a + b = 2(k_a + k_b + 1)$. Thus, by definition $2|(a + b)$, and thus $a + b$ is *even*.

□

Problem 2. Prove that the sum of an *even* and an *odd* number is *odd.*

Answers may vary: I provide two possible solutions. One using a structured, step-by-step derivation, and another streamlined proof written in prose.

Proposition 2. Let $a \in \mathbb{Z}$ be *even* and $b \in \mathbb{Z}$ be *odd*, then $a + b$ is *odd.*

Proof.

1. Let $a \in \mathbb{Z}$ be *even* and $b \in \mathbb{Z}$ be *odd.* (by hypothesis)
2. There is some $k_a \in \mathbb{Z}$ such that $a = 2k_a$. (by 1 & definition of *even*)
3. There is some $k_b \in \mathbb{Z}$ such that $b = 2k_b + 1$. (by 1 & definition of *odd*)
4. $a + b = a + b$. (by reflexivity of $=$)
5. $a + b = (2k_a) + b$. (by 2 & 4)
6. $a + b = (2k_a) + (2k_b + 1)$ (by 3 & 5)
7. $a + b = 2(k_a + k_b) + 1$ (by 6 & simplification)
8. $k_a + k_b$ is an integer (sum of integers is an integer)
9. $a + b$ is *odd.* (by 7, 8, & definition of *odd*)

□

Proof. Let a be an *even* integer and b an *odd* integer. By definition there must be some $k_a \in \mathbb{Z}$ and $k_b \in \mathbb{Z}$ such that $a = 2k_a$ and $b = 2k_b + 1$. Necessarily, $a + b = (2k_a) + (2k_b + 1)$. By simplification, we have $a + b = 2(k_a + k_b) + 1$. Thus, by definition $a + b$ is *odd.* □

Problem 3. Prove that if a is *even* whenever b is *even*, then $a + b$ is *even*.

Answers may vary: I provide two possible solutions. One using a structured, step-by-step derivation, and another streamlined proof written in prose.

Lemma A. If $a \in \mathbb{Z}$ is not *even*, then a is *odd*.

Proof. Every integer can be written in the form of either $2k$ or $2k + 1$ for some integer k. Since a is not *even*, a is not of the form $2k$ for any k. Thus, a must take the form $2k + 1$. Thus, a is *odd*. □

Proposition 3. Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be integers. If a is *even* iff b is even, then $a + b$ is *even*.

$$(a \text{ is } \textit{even} \iff b \text{ is } \textit{even}) \implies a + b \text{ is } \textit{even}$$

Proof.

1. a is *even* iff b is *even* (by hypothesis)
2. Either a and b is *even* or neither a nor b is *even* (by defnition of iff)
3. Assume both a and b are *even* (by 2 & case analysis)
   (a) There is some $k_a \in \mathbb{Z}$ such that $a = 2k_a$. (by 3 & definition of *even*)
   (b) There is some $k_b \in \mathbb{Z}$ such that $b = 2k_b$. (by 3 & definition of *even*)
   (c) $a + b = a + b$. (by reflexivity of =)
   (d) $a + b = (2k_a) + b$. (by 3a & 3c)
   (e) $a + b = (2k_a) + (2k_b)$. (by 3b & 3d)
   (f) $a + b = 2(k_a + k_b)$. (by 3e & simplification)
   (g) $a + b$ is *even*. (by 3f & definition of *even*)
4. Assume both a and b are not *even* (by 2 & case analysis)
   (a) a and b are *odd*. (by 4 & Lemma A)
   (b) $a + b$ is *odd*. (by 4a & Proposition 1)
5. $a + b$ is *even*. (by 3g & 4b)

□

Proof. By assumption, a is *even* if and only if b is *even*. Either both a and b are *even* or neither a nor b is *even*.

Case: Both, a and b are *even*. By definition, there must be some $k_a \in \mathbb{Z}$ and $k_b \in \mathbb{Z}$ such that $a = 2k_a$ and $b = 2k_a$. Thus $a + b = 2k_a + 2k_b$. By simplification, we have $a + b = 2(k_a + k_b)$, and thus by definition $a + b$ is *even*.

Case: Neither a nor b are *even*. By Lemma 1, we may conclude that both a and b are both *odd*. Then by Proposition 1, we may conclude that $a + b$ is *even*.

In both cases, $a + b$ is *even*. Thus we may conclude $a + b$ is *even*. □

Problem 4. Prove for $0 < a \leq b < c$, that if a *divides* b and a *divides* c, then $\frac{c-b}{a}$ is a positive integer.

Answers may vary: I provide two possible solutions. One using a structured, step-by-step derivation, and another streamlined proof written in prose.

Proposition 4. For $0 < a \leq b < c$, if a *divides* b and a *divides* c, then $\frac{c-b}{a}$ is a positive integer.

Proof.

1. a *divides* b. (by assumption)
2. There is a $k_b \in \mathbb{Z}$ such that $b = ak_b$. (by 1 & definition of *divides*)
3. a *divides* c (by assumption)
4. There is a $k_c \in \mathbb{Z}$ such that $c = ak_c$. (by 1 & definition of *divides*)
5. $c - b = c - b$. (by reflexivity of $=$)
6. $c - b = (ak_c) - b$. (by 2 & 5)
7. $c - b = (ak_c) - (ak_b)$. (by 4 & 6)
8. $c - b = a(k_c - k_b)$. (by 7 & simplification)
9. $\frac{c-b}{a} = \frac{a(k_c - k_b)}{a} = k_c - k_b$. (by 8 & simplification)
10. $b < c$. (by assumption)
11. $ak_b < ak_c$. (by 2, 4, & 10)
12. $0 < a$. (by assumption)
13. $k_b < k_c$ (by 11, 12 & order preservation)
14. $0 < k_c - k_b$ (by 13 & definitoin of $<$)
15. $0 < k_c - k_b = \frac{c-b}{a}$ (by 14 & 9)
16. $\frac{c-b}{a}$ is a positive integer (by 15 & definition of positive)

□

Proof. By assumption $0 < a \leq b < c$, a *divides* b, and a *divides* c. By definition, there must be some $k_b \in \mathbb{Z}$ and $k_c \in \mathbb{Z}$ such that $b = ak_b$ and $c = ak_c$. We may then conclude:

$$\frac{c-b}{a} = \frac{ak_c - ak_b}{a} = \frac{a(k_c - k_b)}{a} = k_c - k_b$$

Clearly, $\frac{c-b}{a} = k_c - k_b$ is an integer. Since, $b < c$ we may deduce that $ak_b < ak_c$. Since $0 < a$, we may further conclude that $k_b < k_c$ and thus $0 < k_c - k_b$. By definition, $k_b - k_c$ is positive. Since $\frac{c-b}{a} = k_c - k_b$ we may conclude that $\frac{c-b}{a}$ is a positive integer. □

Problem 5. Prove that if b is *odd* and $a|b$, then a is *odd.*

Answers may vary: I provide two possible solutions. One using a structured, step-by-step derivation, and another streamlined proof written in prose.

We first proceed to prove a more general proposition.
Proposition 5. For integers a and b, if ab is *odd*, then a is *odd.*

Proof. We begin by proving the contrapositive: if a is not *odd*, then ab is not *odd.*
By assumption, a is not *odd.* By (the contrapositive of) Lemma A, we may determine that a is *even.* Since a is *even*, we know there is some k_a such that $a = 2k_a$ and thus $ab = (2k_a)b = 2(k_a b)$. By definition, ab is *even.* By Lemma A, we maythen determine ab is not *odd.* □

Corallary B. If b is *odd* and $a|b$,then a is *odd.*

Proof.

1. b is *odd.* (by assumption)
2. a *divides* b. (by assumption)
3. There is some $k_b \in \mathbb{Z}$ such that $b = ak_b$. (by 2 & definition of *divides*)
4. ak_b is *odd* (by 1 & 3)
5. a is *odd* (by 4 & Proposition 5)

□