Induction: Group Exercises

CSCI 246

March 2, 2026

Problem 1. Prove using either induction / strong induction that:

$$\forall n \in \mathbb{N}.\ \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Proof.

Let n be any natural number. Proof by induction on n:

Base case: $n = 1$.

$$\sum_{k=1}^{n} k = \sum_{k=1}^{1} k = 1 = \frac{2}{2} = \frac{1(1+1)}{2} = \frac{n(n+1)}{2}$$

Inductive Case: $n = m + 1$ and $\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$ (*inductive hypothesis*).

$$\sum_{k=1}^{n} k = \left(\sum_{k=1}^{m} k\right) + (m+1) = \left(\frac{m(m+1)}{2}\right) + (m+1) = \frac{m(m+1)}{2} + \frac{2(m+1)}{2}$$
$$= \frac{m(m+1) + 2(m+1)}{2} = \frac{(m+2)(m+1)}{2} = \frac{((m+1)+1)(m+1)}{2} = \frac{(n+1)n}{2} = \frac{n(n+1)}{2}$$

□

Problem 2. Prove that the sum of the first n odd numbers is n^2:

$$\forall n \in \mathbb{N}.\ \sum_{k=1}^{n} 2k - 1 = n^2$$

Proof.

Let n be any natural number. Proof by induction on n:

Base Case: $n = 1$

$$\sum_{k=1}^{n} 2k - 1 = \sum_{k=1}^{1} 2k - 1 = 2(1) - 1 = 1 = 1^2 = n^2$$

Inductive Case: $n = m + 1$ and $\sum_{k=1}^{m} 2k + 1 = m^2$

$$\sum_{k=1}^{n} 2k - 1 = \sum_{k=1}^{m+1} 2k - 1 = \left(\sum_{k=1}^{m} 2k - 1\right) + 2(m+1) - 1 = m^2 + 2m + 1 = (m+1)^2 = n^2$$

□

Problem 3. Let $b \in \mathbb{B}^n$ be an n-bit binary number. Prove that the maximum value of b is at most $2^n - 1$.

Proof.

An n-bit binary number takes its maximum value when each of the n bits are set.
The value of such a binary number is $\sum_{k=0}^{n-1} 2^k$. Thus it suffices to prove forall $n \in \mathbb{N}_0$ that:

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1$$

We proceed by proof by induction on n.
Base Case: $n = 0$

$$\sum_{k=0}^{n-1} 2^k = \sum_{k=0}^{0-1} 2^k = \sum_{k=0}^{-1} 2^k = 0 = 1 - 1 = 2^0 - 1 = 2^n - 1$$

Inductive Case: $n = m + 1$ and $\sum_{k=0}^{m-1} 2^k = 2^m + 1$

$$\sum_{k=0}^{n-1} 2^k = \sum_{k=0}^{(m+1)-1} 2^k = \sum_{k=0}^{m} 2^k = \left(\sum_{k=0}^{m-1} 2^k\right) + 2^m = (2^m - 1) + 2^m = 2(2^m) - 1 = 2^{m+1} - 1 = 2^n - 1$$

$\square$

Problem 4. Prove for any natural number that $2^n \geq n + 1$.

Proof.

Let n be any natural number. Proof by induction on n:
Base Case: $n = 1$
Clearly, $2^n = 2^0 = 1 \geq 1 = 0 + 1 = n + 1$, and thus $2^n \geq n + 1$ for $n = 0$.

Inductive Case: $n = m + 1$ and $2^m \geq m + 1$
By assumption $2^n = 2^{m+1} = 2(2^m)$.
Similarly, $n + 1 = (m + 1) + 1 = m + 2$.
By the inductive hypothesis we have $2^m \geq m + 1$.
Since $m \geq 0$, we also know $2(2^m) \geq 2(m + 1) = 2m + 2$.
Similarly, $2m + 2 = m + (m + 2) \geq m + 2$. To summarize:

$$2^n = 2(2^m) \geq 2m + 2 \geq m + 2 = n + 1$$

Thus, we may conclude that $2^n \geq n + 1$. $\square$

Problem 5. Prove for any $n \in \mathbb{N}$ that $3|(n^3 - n)$.

Proof.

Let n be any natural number.

Base Case: $n = 1$

Clearly, $3|0$ and $0 = 1 - 1 = 1^3 - 1 = n^3 - n$. Thus $3|(n^3 - n)$ for $n = 1$.

Inductive Case: $n = m + 1$ and $3|(m^3 - m)$

By substitution and algebraic manipulation we have:

$$n^3 - n = (m+1)^3 - (m+1) = m^3 + 3m^2 + 3m + 1 - m - 1 = (m^3 - m) + 3(m^2 + m)$$

By the inductive hypothesis, we know that $3|(m^3 - m)$.
Thus there must be some k such that $m^3 - m = 3k$.
Therefore $n^3 - n = 3k + 3(m^2 + m) = 3(k + m^2 + m)$.
Clearly, $3|3(k + m^2 + m)$. Thus $3|(n^3 - n)$ as required. □

Problem 6. Let F_n the nth Fibonacci number with $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$—i.e., the nth Fibonacci number is the sum of the two preceeding Fibonacci numbers. Prove that $F_n < 2^n$ for any natural number $n > 0$.

Proof.

Let $n \in \mathbb{N}$ be any natural number (greater than 0). Proof by strong induction.

Base Case: $n = 1$.

Clearly, $F_n = F_1 = 1 < 2 = 2^1 = 2^n$. Thus, $F_n < 2^n$ for $n = 1$.

Base Case: $n = 2$.

Necessarily, $F_n = F_2 = F_1 + F_0 = 1 + 1 = 2 < 4 = 2^2 = 2^n$. Thus, $F_n < 2^n$ for $n = 2$.

Inductive Case: $2 < n$ and $\forall m \in \mathbb{N}_0.\ 0 < m < n \Rightarrow F_m < 2^m$.

By assumption $2 < n$, and thus we have $F_n = F_{n-1} + F_{n-2}$.
Necessarily, $n - 1 > 0$ and $n - 2 > 0$.
Thus we may apply our inductive hypothesis to get $F_{n-1} < 2^{n-1}$ and $F_{n-1} < 2^{n-2}$.
Therefore we know that $F_n = F_{n-1} + F_{n-2} < 2^{n-1} + 2^{n-2} = 2(2^{n-2}) + 2^{n-2} = 3(2^{n-2})$.
Next, note that $2^n = 4(2^{n-2}) > 3(2^{n-2})$.
To summarise, $F_n = F_{n-1} + F_{n-2} < 3(2^{n-2}) < 4(2^{n-2}) = 2^n$.
Thus, we may conclude that $F_n < 2^n$. □