Recurrence Relations: Group Exercises

CSCI 246

March 4, 2026

Problem 1. Consider the following recurrence relation $a_n = 3a_{n-1}$ with $a_0 = 2$.

A. Compute a_1, a_2, a_3, a_4, and a_5.

$$\begin{array}{rcrcrcr} a_0 & = & 2 & & & = & 2 \\ a_1 & = & 3a_0 & = & 3(2) & = & 6 \\ a_2 & = & 3a_1 & = & 3(6) & = & 18 \\ a_3 & = & 3a_2 & = & 3(18) & = & 54 \\ a_4 & = & 3a_3 & = & 3(54) & = & 162 \\ a_5 & = & 3a_4 & = & 3(162) & = & 486 \end{array}$$

B. Based on the computed values, conjecture a closed form solution to a_n.

$$a_n = 2(3^n)$$

C. Solve for the closed form solution of a_n.

For any recurrence relation of the form $a_n = sa_{n-1}$ we know the closed form is $a_n = (a_0)s^n$. Since $a_n = 3a_{n-1}$ follows this form and $a_0 = 2$, we know that $a_n = 2(3^n)$.

Problem 2. Consider the following recurrence relation $a_n = 2a_{n-1} + 1$ and $a_0 = 1$.

A. Compute a_1, a_2, a_3, a_4, and a_5.

$$\begin{array}{rcrcrcr} a_0 & = & 1 & & & = & 1 \\ a_1 & = & 2a_0 + 1 & = & 2(1) + 1 & = & 3 \\ a_2 & = & 2a_1 + 1 & = & 2(3) + 1 & = & 7 \\ a_3 & = & 2a_2 + 1 & = & 2(7) + 1 & = & 15 \\ a_4 & = & 2a_3 + 1 & = & 2(15) + 1 & = & 31 \\ a_5 & = & 2a_4 + 1 & = & 2(31) + 1 & = & 63 \end{array}$$

B. Based on the computed values, conjecture a closed form solution to a_n.

$$a_n = 2^{n+1} - 1$$

C. Solve for the closed form solution of a_n.

For any recurrence relation of the form $a_n = sa_{n-1} + t$, we know the closed form is $a_n = C_1 s^n + C_2$ for some constants C_1 and C_2. We can setup and solve the following linear equations to find C_1 and C_2.

$$\begin{array}{rcrcl} a_0 & = & 1 & = & C_1 s^0 + C_2 \\ a_1 & = & 3 & = & C_1 s^1 + C_2 \end{array} \qquad \begin{array}{rcr} 1 & = & C_1 + C_2 \\ 3 & = & 2C_1 + C_2 \end{array} \qquad \begin{array}{rcr} C_1 & = & 2 \\ C_2 & = & -1 \end{array}$$

Thus $a_n = 2(2^n) - 1 = 2^{n+1} - 1$.

Problem 3. Consider the following recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ with $a_0 = 1$ and $a_1 = 5$.

A. Compute a_2, a_3, a_4, and a_5.

$$\begin{array}{lllllll} a_0 & = & 1 & & & = & 1 \\ a_1 & = & 5 & & & = & 5 \\ a_2 & = & 5a_1 - 6a_0 & = & 5(5) - 6(1) & = & 19 \\ a_3 & = & 5a_2 - 6a_1 & = & 5(19) - 6(5) & = & 65 \\ a_4 & = & 5a_3 - 6a_2 & = & 5(65) - 6(19) & = & 211 \\ a_5 & = & 5a_4 - 6a_3 & = & 5(211) - 6(65) & = & 665 \end{array}$$

B. Based on the computed values, conjecture a closed form solution to a_n.

$a_n = 3^{n+1} - 2^{n+1}$ honestly, I didn't have a guess beyond "some exponential"

C. Solve for the closed form solution of a_n.

Since a_n is a second order recurrence relation of the form $a_n = sa_{n-1} + ta_{n-2}$ we must first find the roots of a_n's characteristic polynomial $r^2 - sr - t = 0$. The roots of $r^2 - 5r + 6 = (r-2)(r-3)$ are 2 and 3. Since the characteristic polynomial has two distinct roots, we know the closed form solution of a_n is $a_n = C_1(2^n) + C_2(3^n)$ for some constants C_1 and C_2. We can setup and solve the following linear equations to find the value for C_1 and C_2.

$$\begin{array}{lllll} a_0 & = & 1 & = & C_1(2^0) + C_2(3^0) \\ a_1 & = & 5 & = & C_1(2^1) + C_2(3^1) \end{array} \qquad \begin{array}{lll} 1 & = & C_1 + C_2 \\ 5 & = & 2C_1 + 3C_2 \end{array} \qquad \begin{array}{lll} C_1 & = & -2 \\ C_2 & = & 3 \end{array}$$

Thus $a_n = 3(3^n) - 2(2^n) = 3^{n+1} - 2^{n+1}$.

Problem 4. Consider the following recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$ with $a_0 = 2$ and $a_1 = 5$.

A. Compute a_2, a_3, a_4, and a_5.

$$\begin{array}{lllllll} a_0 & = & 2 & & & = & 2 \\ a_1 & = & 5 & & & = & 5 \\ a_2 & = & 4a_1 - 3a_0 & = & 4(5) - 3(2) & = & 14 \\ a_3 & = & 4a_2 - 3a_1 & = & 4(14) - 3(5) & = & 41 \\ a_4 & = & 4a_3 - 3a_2 & = & 4(41) - 3(14) & = & 122 \\ a_5 & = & 4a_4 - 3a_3 & = & 4(122) - 3(41) & = & 365 \end{array}$$

B. Based on the computed values, conjecture a closed form solution to a_n.

$a_n = \dfrac{3^{n+1} + 1}{2}$ honestly, I didn't have a guess beyond "some exponential"

C. Solve for the closed form solution of a_n.

Since a_n is a second order recurrence relation of the form $a_n = sa_{n-1} + ta_{n-2}$ we must first find the roots of a_n's characteristic polynomial $r^2 - sr - t = 0$. The roots of $r^2 - 4r + 3 = (r-1)(r-3)$ are 1 and 3. Since the characteristic polynomial has two distinct roots, we know the closed form solution of a_n is $a_n = C_1(1^n) + C_2(3^n)$ for some constants C_1 and C_2. We can setup and solve the following linear equations to find the value for C_1 and C_2.

$$\begin{array}{lllll} a_0 & = & 2 & = & C_1(1^0) + C_2(3^0) \\ a_1 & = & 5 & = & C_1(1^1) + C_2(3^1) \end{array} \qquad \begin{array}{lll} 2 & = & C_1 + C_2 \\ 5 & = & C_1 + 3C_2 \end{array} \qquad \begin{array}{lll} C_1 & = & \frac{1}{2} \\ C_2 & = & \frac{3}{2} \end{array}$$

Thus $a_n = \frac{3}{2}(3^n) + \frac{1}{2}(1^n) = \frac{3^{n+1}+1}{2}$.

Problem 5. A freelance programmer starts with \$1000 in savings. Each month interest compounds her savings by 5% and she deposits an additional \$200. Write a recurrence relation that captures how much she has saved by month n. Then solve for how much she will have saved after 5 years (n = 60).

Let a_n represent the amount of money she has saved at the end of month n be a_n. Initially she has \$1000. Saved. Thus $a_0 = 1000$. Since she adds \$200 a month and interest is 5%. The general recurrence relation is $a_n = 1.05a_{n-1} + 200$. To solve the recurrence relation, we know that the closed form solution is of the form $a_n = C_1(1.05^n) + C_2$ for some constants C_1 and C_2.

$$\begin{array}{rclcl} a_0 &=& 1000 &=& C_1(1.05^0) + C_2 \\ a_1 &=& 1250 &=& C_1(1.05^1) + C_2 \end{array} \qquad \begin{array}{rcr} 1000 &=& C_1 + C_2 \\ 1250 &=& 1.05C_1 + C_2 \end{array} \qquad \begin{array}{rcr} C_1 &=& 5000 \\ C_2 &=& -4000 \end{array}$$

Thus the closed form solution is $a_n = 5000(1.05^n) - 4000$. After 5 years, the freelancer has saved a total of \$89,395.93 (rounding to nearest cent, since $a_{60} = 5000(1.05^{60}) - 4000 = 89395.9295$).

Problem 6. Suppose you operate a distributed system that handles a number of requests each hour (at time t). Over time the number of requests grow. You notice that you gain three times as many requests from last hour but lose three times as many requests from two hours ago. Write a recurrence relation that captures the number of requests received at time period t. Assuming that $a_0 = 1$ and $a_1 = 4$, what is the number of requests processed after 100 hours of operation? How fast is the number of requests growing?

Let a_n represent the number of requests received during hour n. Since the number of requests grows by three times the previous period but we lose three times as many requests from two hours ago, we have $a_n - a_{n-1} = 3a_{n-1} - 3a_{n-2}$ and thus $a_n = 4a_{n-1} - 3a_{n-2}$. We are given, $a_0 = 1$ and $a_1 = 4$. Since a_n is a second order recurrence relation we must find the roots of its characteristic polynomial to determine the closed form solution of a_n. The characteristic polynomial is $r^2 - 4r + 3 = 0 = (r-3)(r-1)$ (i.e., it has roots $r = 1, 3$). Since it has two distinct roots, we know $a_n = C_1(1^n) + C_2(3^n)$ for some constants C_1 and C_2. Now, we setup and solve linear equations (using a_0 and a_1) to determine C_1 and C_2.

$$\begin{array}{rclcl} a_0 &=& 1 &=& C_1(1^0) + C_2(3^0) \\ a_1 &=& 4 &=& C_1(1^1) + C_2(3^1) \end{array} \qquad \begin{array}{rcl} 1 &=& C_1 + C_2 \\ 4 &=& C_1 + 3C_2 \end{array} \qquad \begin{array}{rcr} C_1 &=& -\frac{1}{2} \\ C_2 &=& \frac{3}{2} \end{array}$$

Thus $a_n = \frac{3}{2}(3^n) - \frac{1}{2}(1^n) = \frac{3^{n+1}-1}{2}$. Asymptotically, as n grows larger, the number of requests triples every hour. The number of requests received in the 100th hour is $a_{100} = \frac{3^{101}-1}{2} \sim 1.546 \times 10^{48}$

Problem 7. Suppose you are running a startup. You notice each month the number of users triples; however, due to market saturation growth slows proportional to the previous month's growth. After modeling, you find that the number of users in month n is $a_n = 6a_{n-1} - 9a_{n-2}$. Show that the growth is not purely exponential.

Since the number of users is described by the second-order recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$, we need to solve for the roots of the characteristic polynomial of the recurrence relation to determine the closed form of a_n. The characteristic polynomial is $r^2 - 6r + 9 = 0 = (r-3)(r-3)$ (i.e., it has a single root $r = 3$ with multiplicity 2). Since it has a single root, we know $a_n = (C_1 + C_2n)3^n$ for some constants C_1 and C_2.

In general, the growth is an exponential (3^n) times a linear term ($C_1 + C_2n$). Since we do not know the initial number of users, we cannot solve for the exact value of C_1 and C_2. However, since C_2 is multiplied by n, we know that $C_1 = a_0$, and then we can derive that $C_2 = \frac{a_1 - 3a_0}{3}$. Thus, the linear term is $a_0 + \frac{a_1 - 3a_1}{3}n$. We can examine the cases $a_1 < 3a_0$, $a_1 = 3a_0$, or $a_1 > 3a_0$ (i.e., if the term multiplied by n is negative, zero, or positive). If it's negative, then the number of users will initially grow and then eventually keep decreasing forever (or realistically until 0 since one cannot have negative users). If the term is 0 then the number of users will grow exponentially (triple each month). If the term is positive, the number of users will grow super-exponentially as n grows.