Pumping Lemma
CSCI 338
Claim: The language $L = \{0^n1^n : n \geq 0\}$ is not regular.

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Proof: Suppose $L$ is regular.
Pumping Lemma Example 1

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Proof: Suppose $L$ is regular.

Then, $L$ must abide by the pumping lemma.

\[
\text{Pumping Lemma: Given a regular language } L, \exists \text{ a number } p \text{ such that any string } s \in L, \text{ with } |s| \geq p, \text{ can be divided into three pieces, } s = xyz \text{ satisfying:}
\]
\begin{enumerate}
  \item $xy^i z \in L, \forall i \geq 0$.
  \item $|y| > 0$.
  \item $|xy| \leq p$.
\end{enumerate}
Pumping Lemma Example 1

Claim: The language $L = \{0^n1^n: n \geq 0\}$ is not regular.

Proof: Suppose $L$ is regular.
Then, $L$ must abide by the pumping lemma. I.e. There must be some number $p$ ...

Pumping Lemma: Given a regular language $L$, $\exists$ a number $p$ such that any string $s \in L$, with $|s| \geq p$, can be divided into three pieces, $s = xyz$ satisfying:

1. $xy^iz \in L, \forall i \geq 0$.
2. $|y| > 0$.
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Pumping Lemma: Given a regular language \( L \), \( \exists \) a number \( p \) such that any string \( s \in L \), with \( |s| \geq p \), can be divided into three pieces, \( s = xyz \) satisfying:

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Claim: The language $L = \{0^n1^n: n \geq 0\}$ is not regular.

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Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma.

Now, we need some appropriate string ($s \in L$ and $|s| \geq p$) that will break condition 1 when we allow multiple $y$’s.

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1. $xy^i z \in L, \forall i \geq 0$.
2. $|y| > 0$.
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Claim: The language \( L = \{0^n1^n : n \geq 0\} \) is not regular.

Proof: Suppose \( L \) is regular. Let \( p \) be the number from the pumping lemma.

Consider \( s = 0^p1^p \).

Since \( s \in L \) and \( |s| \geq p \), the conditions of the pumping lemma must hold for \( s = xyz \).

**Pumping Lemma**: Given a regular language \( L \), \( \exists \) a number \( p \) such that any string \( s \in L \), with \( |s| \geq p \), can be divided into three pieces, \( s = xyz \) satisfying:

1. \( xy^iz \in L \) for all \( i \geq 0 \).
2. \( |y| > 0 \).
3. \( |xy| \leq p \).
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Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma.

Consider $s = 0^p1^p$.

Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

To break condition 1, we need to learn more about $y$

**Pumping Lemma:** Given a regular language $L$, $\exists$ a number $p$ such that any string $s \in L$, with $|s| \geq p$, can be divided into three pieces, $s = xyz$ satisfying:

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Consider $s = 0^p1^p$.
Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

$y = 0^k$ for some $k > 0$, since

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Consider $s = 0^p1^p$.
Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.
$y = 0^k$ for some $k > 0$,
since $|xy| \leq p$

$$s = \underbrace{000 \ldots}_{p} \underbrace{0011 \ldots}_{p} 111$$
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\[
\begin{align*}
  s &= 0|00|...0011...111 \\
  x &= p \\
  y &= p \\
  z &= \underbrace{00|...0011...111}_{p}
\end{align*}
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s = \underbrace{000\ldots0|011\ldots111}_{p} \underbrace{x\underbrace{011\ldots111}_{p}}_{y}\underbrace{011\ldots111}_{z}
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  s &= \underbrace{000 \ldots 00}_{p} \underbrace{11 \ldots 11}_{p} \\
  &= xy \underbrace{11 \ldots 11}_{p} \\
  &= x \underbrace{00 \ldots 00}_{p} \underbrace{11 \ldots 11}_{p}
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\textbf{s} &= \underbrace{000 \ldots 001}_{p} |1 \ldots 111_{p} \\
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\[ s = \overbrace{000 \ldots 0011 \ldots 111}^{p} \]

\[ x \quad y \quad z \]

\[ |xy| = p + 1 > p \]
Claim: The language $L = \{0^n1^n : n \geq 0\}$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma.
Consider $s = 0^p1^p$.

Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

$y = 0^k$ for some $k > 0$,
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There is no possible way to partition $s$ into $xyz$ where both:

1. $|xy| \leq p$
2. $y$ has 1s in it.

$$s = \underbrace{000 \ldots 0011 \ldots 111}_x \underbrace{00 \ldots 00}_{y} \underbrace{11 \ldots 11}_{z}$$

$|xy| = p + 1 > p$
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Proof: Suppose \( L \) is regular. Let \( p \) be the number from the pumping lemma.

Consider \( s = 0^p1^p \).

Since \( s \in L \) and \( |s| \geq p \), the conditions of the pumping lemma must hold for \( s = xyz \).

\( y = 0^k \) for some \( k > 0 \), since \( |xy| \leq p \).

Since \( |xy| \leq p \), \( y \) must be in the first \( p \) characters of every string. Since the first \( p \) characters of this string are all 0, \( y \) must contain all 0s.

Pumping Lemma: Given a regular language \( L \), \( \exists \) a number \( p \) such that any string \( s \in L \), with \( |s| \geq p \), can be divided into three pieces, \( s = xyz \) satisfying:

1. \( xy^iz \in L, \forall i \geq 0 \).
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Claim: The language $L = \{0^n1^n: n \geq 0\}$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma.
Consider $s = 0^p1^p$.
Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.
$y = 0^k$ for some $k > 0$, since $|xy| \leq p$
$\Rightarrow s = 0^{p-k}0^k1^p$
Claim: The language $L = \{0^n1^n: n \geq 0\}$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma.

Consider $s = 0^p1^p$.

Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

$y = 0^k$ for some $k > 0$, since $|xy| \leq p$

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Consider the string $s' = xy^2z$
Pumping Lemma Example 1

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Consider $s = 0^p1^p$.

Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

$y = 0^k$ for some $k > 0$, since $|xy| \leq p$

$\implies s = 0^{p-k}0^k1^p$

Consider the string $s' = xy^2z = 0^{p-k}0^{2k}1^p$
Pumping Lemma Example 1

Claim: The language \(L = \{0^n1^n : n \geq 0\}\) is not regular.

Proof: Suppose \(L\) is regular. Let \(p\) be the number from the pumping lemma. Consider \(s = 0^p1^p\).

Since \(s \in L\) and \(|s| \geq p\), the conditions of the pumping lemma must hold for \(s = xyz\). 

\(y = 0^k\) for some \(k > 0\), since \(|xy| \leq p\)

\[\Rightarrow s = 0^{p-k}0^k1^p\]

Consider the string \(s' = xy^2z = 0^{p-k}0^{2k}1^p\)

\[\Rightarrow s' \notin L.\]
Pumping Lemma Example 1

**Claim:** The language \( L = \{0^n1^n: n \geq 0\} \) is not regular.

**Proof:** Suppose \( L \) is regular. Let \( p \) be the number from the pumping lemma.

Consider \( s = 0^p1^p \).

Since \( s \in L \) and \( |s| \geq p \), the conditions of the pumping lemma must hold for \( s = xyz \).

\( y = 0^k \) for some \( k > 0 \), since \( |xy| \leq p \)
\( \Rightarrow s = 0^{p-k}0^k1^p \)

Consider the string \( s' = xy^2z = 0^{p-k}0^{2k}1^p \)

But, the number of zeros = \( p - k + 2k = p + k \) > number of ones = \( p \).
\( \Rightarrow s' \notin L \).
Claim: The language $L = \{0^n1^n : n \geq 0\}$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma. Consider $s = 0^p1^p$. Since $s \in L$ and $s \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

$y = 0^k$ for some $k > 0$, $\Rightarrow s = 0^{p-k}0^k1^p$.

Consider the string $s' = x'y'z = 0^{p-k}0^k1^{p-k}$.

But, the number of zeros $= p - k + 2k = p + k >$ number of ones $= p$.

$\Rightarrow s' \notin L$. But the pumping lemma said this should work for every string in $L$!
Pumping Lemma Example 1

Claim: The language $L = \{0^n1^n: n \geq 0\}$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma.

Consider $s = 0^p1^p$.

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Consider the string $s' = xy^2z = 0^{p-k}0^{2k}1^p$

But, the number of zeros $= p - k + 2k = p + k >$ number of ones $= p$.

$\Rightarrow s' \not\in L$, which is a contradiction of the pumping lemma.
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Claim: The language \( L = \{0^n1^n: n \geq 0\} \) is not regular.

Proof: Suppose \( L \) is regular. Let \( p \) be the number from the pumping lemma.

Consider \( s = 0^p1^p \).

Since \( s \in L \) and \( |s| \geq p \), the conditions of the pumping lemma must hold for \( s = xyz \).

\[ y = 0^k \text{ for some } k > 0, \text{ since } |xy| \leq p \]

\[ \Rightarrow s = 0^{p-k}0^k1^p \]

Consider the string \( s' = xy^2z = 0^{p-k}0^{2k}1^p \)

But, the number of zeros \( = p - k + 2k = p + k \) > number of ones \( = p \).

\[ \Rightarrow s' \notin L, \text{ which is a contradiction of the pumping lemma.} \]

Therefore, the language is not regular.
DFA/NFA Limitations

\[ \{0^n1^n : n \geq 0\} \]

All Languages

Regular Languages
Pumping Lemma Proof Blueprint

Claim: Some language $L$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma.

Consider $s = ?$. Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

Consider the string $s' = xy^2z = ?$

$\Rightarrow s' \notin L$, which is a contradiction of the pumping lemma.

Therefore, the language is not regular.
Pumping Lemma Proof Blueprint

Claim: Some language $L$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma. 
Consider $s = ?$. \textbf{1 - Select $s$ that will work with $s \in L$ and $|s| \geq p$} 
Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$. 

Consider the string $s' = xy^2z = ?$

$\Rightarrow s' \notin L$, which is a contradiction of the pumping lemma.
Therefore, the language is not regular.
Pumping Lemma Proof Blueprint

Claim: Some language $L$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma.

Consider $s = ?$. 1 – **Select s that will work with $s \in L$ and $|s| \geq p$**

Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

? 2 – **Find some conditions that $y$ must meet**

Consider the string $s' = x y^2 z = ?$

? 

$\Rightarrow s' \notin L$, which is a contradiction of the pumping lemma.

Therefore, the language is not regular.
Pumping Lemma Proof Blueprint

Claim: Some language $L$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma.

Consider $s = \ ?$. 1 – Select $s$ that will work with $s \in L$ and $|s| \geq p$

Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

2 – Find some conditions that $y$ must meet

Consider the string $s' = x y^i z = \ ?$

$\Rightarrow s' \notin L$, which is a contradiction of the pumping lemma.

Therefore, the language is not regular.
Pumping Lemma Proof Blueprint

Claim: Some language $L$ is not regular.

Proof: Suppose $L$ is regular. Let $p$ be the number from the pumping lemma. Consider $s = \square$. Since $s \in L$ and $|s| \geq p$, the conditions of the pumping lemma must hold for $s = xyz$.

1 – Select $s$ that will work with $s \in L$ and $|s| \geq p$

2 – Find some conditions that $y$ must meet

3 – Select an $i$ (number of times to repeat $y$)

Consider the string $s' = xy^iz = \square$.

4 – Show what $s'$ equals

$\Rightarrow s' \notin L$, which is a contradiction of the pumping lemma.

Therefore, the language is not regular.
Pumping Lemma Proof Blueprint

Claim: Some language \( L \) is not regular.

Proof: Suppose \( L \) is regular. Let \( p \) be the number from the pumping lemma.

Consider \( s = \)?.  
1 – Select \( s \) that will work with \( s \in L \) and \( |s| \geq p \)

Since \( s \in L \) and \( |s| \geq p \), the conditions of the pumping lemma must hold for \( s = xyz \).

2 – Find some conditions that \( y \) must meet

Consider the string \( s' = x y^i z = \)?  
3 – Select an \( i \) (number of times to repeat \( y \))

\( 4 – \) Show what \( s' \) equals

5 – Show \( s' \) is not in \( L \)

\( \Rightarrow s' \notin L \), which is a contradiction of the pumping lemma.

Therefore, the language is not regular.