Undecidability

CSCI 338

A_{TM}

Claim: $A_{TM} = \{\langle M, \omega\rangle : M$ is a TM and M accepts ω} is undecidable.

Proof: Suppose A_{TM} is decidable. Let TM H be its decider:

$$H(\langle M, \omega\rangle) = \begin{cases} \text{accept, if } M \text{ accepts } \omega \\ \text{reject, if } M \text{ does not accept } \omega \end{cases}$$

Make a new TM D:

D = on input $\langle N\rangle$, for TM N

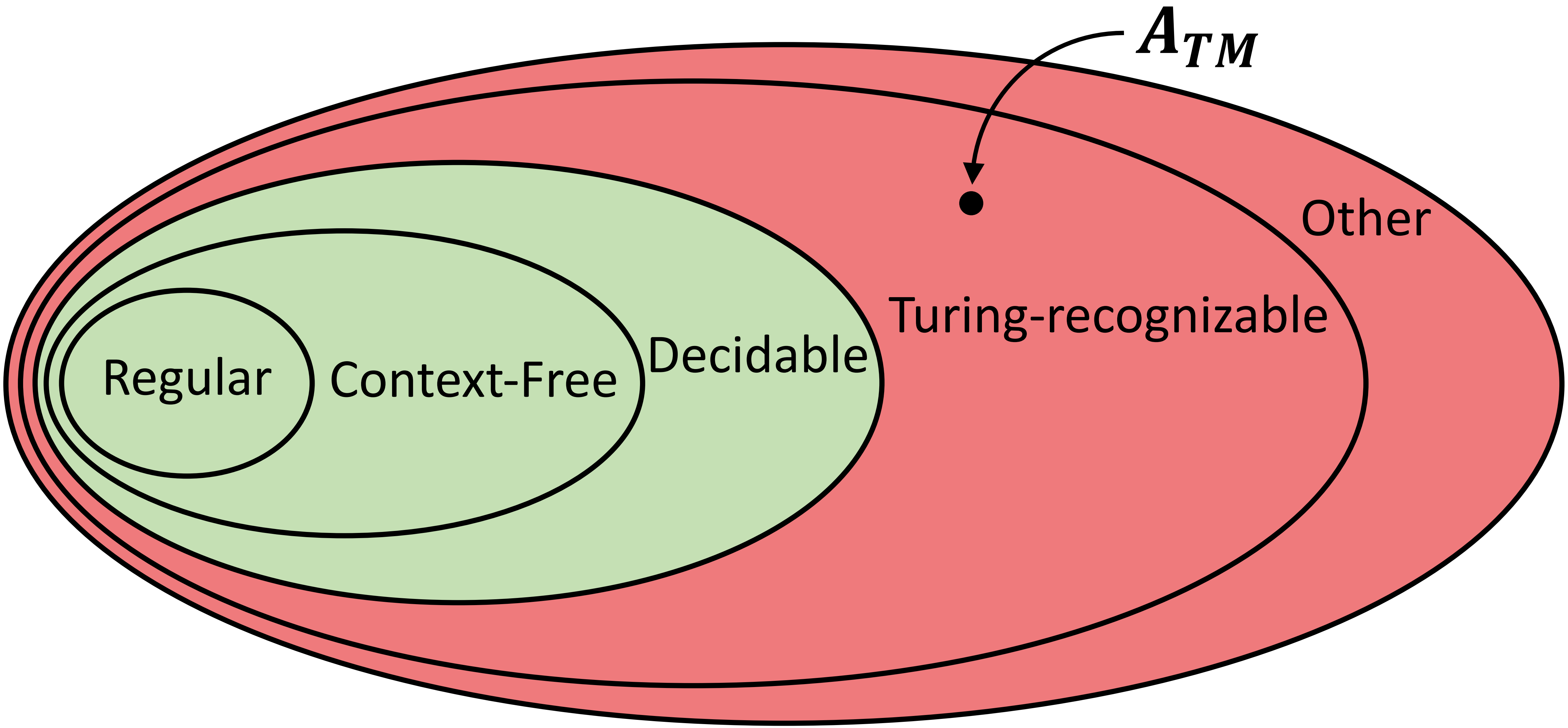
1. Run H on $\langle N, \langle N\rangle\rangle$.
2. If H accepts, <u>reject</u>. If H rejects, <u>accept</u>.

What happens with $D(\langle D\rangle)$? $D(\langle D\rangle) = \begin{cases} \text{accept, if } D \text{ does not accept } \langle D\rangle \\ \text{reject, if } D \text{ accepts } \langle D\rangle \end{cases}$

D accepts $\langle D\rangle$, so long as D does not accept $\langle D\rangle$.

⇒ TM D cannot exist ⟹ TM H cannot exist ⟹ A_{TM} is undecidable

Computability Hierarchy



Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega\rangle : M$ is a TM and M halts on ω$\}$ is undecidable.

Proof:

Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega \rangle : M$ is a TM and M halts on $\omega\}$ is undecidable.

Proof:

We are going to show that a decider for $HALT_{TM}$ can be used to build a decider for A_{TM}.

$A_{TM} = \{\langle M, \omega \rangle : M$ is a TM and M accepts $\omega\}$

Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega \rangle : M$ is a TM and M halts on ω} is undecidable.

Proof: Suppose $HALT_{TM}$ is decidable and let TM H be its decider.

$A_{TM} = \{\langle M, \omega \rangle : M$ is a TM and M accepts ω}

Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega\rangle: M$ is a TM and M halts on ω} is undecidable.

Proof: Suppose $HALT_{TM}$ is decidable and let TM H be its decider.

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega\rangle$

1.

$A_{TM} = \{\langle M, \omega\rangle: M$ is a TM and M accepts ω}

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Proof: Suppose $HALT_{TM}$ is decidable and let TM H be its decider.

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega\rangle$

1. Run H on $\langle M, \omega\rangle$.

$A_{TM} = \{\langle M, \omega\rangle : M$ is a TM and M accepts ω}

Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega\rangle : M$ is a TM and M halts on ω} is undecidable.

Proof: Suppose $HALT_{TM}$ is decidable and let TM H be its decider.

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega\rangle$

1. Run H on $\langle M, \omega\rangle$.
2. If H rejects, <u>reject</u> (i.e. M does not halt on ω).

$A_{TM} = \{\langle M, \omega\rangle : M$ is a TM and M accepts ω}

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S = on input $\langle M, \omega \rangle$

1. Run H on $\langle M, \omega \rangle$.
2. If H rejects, <u>reject</u> (i.e. M does not halt on ω).
3. If H accepts, run M on ω until it halts.

$A_{TM} = \{\langle M, \omega \rangle : M$ is a TM and M accepts $\omega\}$

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4. If M accepts, <u>accept</u>. If M rejects, <u>reject</u>.

$A_{TM} = \{\langle M, \omega\rangle : M$ is a TM and M accepts ω}

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4. If M accepts, <u>accept</u>. If M rejects, <u>reject</u>.

If M accepts ω, S accepts. If M does not accept ω, S rejects. Thus, S is a decider for A_{TM}, which is a contradiction, so $HALT_{TM}$ is undecidable.

Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega\rangle : M$ is a TM and M halts on ω} is undecidable.

Proof: **Suppose $HALT_{TM}$ is decidable and let TM H be its decider.**

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega\rangle$

1. Run H on $\langle M, \omega\rangle$.
2. If H rejects, reject (i.e. M does not halt on ω).
3. If H accepts, run M on ω until it halts.
4. If M accepts, accept. If M rejects, reject.

S is a decider for A_{TM}, which is a contradiction.

$\therefore HALT_{TM}$ is undecidable.

There is sort of a undecidability proof “blueprint”, but it is not as helpful.

Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega\rangle : M$ is a TM and M halts on $\omega\}$ is undecidable.

New problem

Proof: **Suppose $HALT_{TM}$ is decidable and let TM H be its decider.**

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega\rangle$

1. Run H on $\langle M, \omega\rangle$.
2. If H rejects, reject (i.e. M does not halt on ω).
3. If H accepts, run M on ω until it halts.
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S is a decider for A_{TM}, which is a contradiction.

$\therefore$ $HALT_{TM}$ is undecidable.

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Claim: $HALT_{TM} = \{\langle M, \omega \rangle : M$ is a TM and M halts on ω} is undecidable.

New problem

Proof: **Suppose $HALT_{TM}$ is decidable and let TM H be its decider.**

Known undecidable problem

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega \rangle$

1. Run H on $\langle M, \omega \rangle$.
2. If H rejects, reject (i.e. M does not halt on ω).
3. If H accepts, run M on ω until it halts.
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S is a decider for A_{TM}, which is a contradiction.

$\therefore HALT_{TM}$ is undecidable.

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Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega \rangle : M$ is a TM and M halts on ω} is undecidable.

New problem

Proof: Suppose $HALT_{TM}$ is decidable and let TM H be its decider.

Known undecidable problem

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega \rangle$

Input to known problem

1. Run H on $\langle M, \omega \rangle$.
2. If H rejects, reject (i.e. M does not halt on ω).
3. If H accepts, run M on ω until it halts.
4. If M accepts, accept. If M rejects, reject.

S is a decider for A_{TM}, which is a contradiction.

$\therefore HALT_{TM}$ is undecidable.

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Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega \rangle : M$ is a TM and M halts on $\omega\}$ is undecidable.

New problem

Proof: Suppose $HALT_{TM}$ is decidable and let TM H be its decider.

Known undecidable problem

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega \rangle$

Input to known problem

1. Run H on $\langle M, \omega \rangle$.
2. If H rejects, reject (i.e. M does not halt on ω).
3. If H accepts, run M on ω until it halts.
4. If M accepts, accept. If M rejects, reject.

Decider for known problem

S is a decider for A_{TM}, which is a contradiction.

$\therefore HALT_{TM}$ is undecidable.

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Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega\rangle : M$ is a TM and M halts on ω} is undecidable.

Proof: Suppose $HALT_{TM}$ is decidable and let TM H be its decider.

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega\rangle$

1. Run H on $\langle M, \omega\rangle$.
2. If H rejects, <u>reject</u> (i.e. M does not halt on ω).
3. If H accepts, run M on ω until it halts.
4. If M accepts, <u>accept</u>. If M rejects, <u>reject</u>.

If M accepts ω, S accepts. If M does not accept ω, S rejects. Thus, S is a decider for A_{TM}, which is a contradiction, so $HALT_{TM}$ is undecidable.

Halting Problem

Claim: $HALT_{TM} = \{\langle M, \omega \rangle : M$ is a TM and M halts on ω} is undecidable.

Proof: Suppose $HALT_{TM}$ is decidable and let TM H be its decider.

Build a TM S that decides A_{TM}:

S = on input $\langle M, \omega \rangle$

Used $HALT_{TM}$ decider to make decider for A_{TM}
DID NOT use A_{TM} decider to make decider for $HALT_{TM}$

3. If H accepts, run M on ω until it halts.
4. If M accepts, accept. If M rejects, reject.

If M accepts ω, S accepts. If M does not accept ω, S rejects. Thus, S is a decider for A_{TM}, which is a contradiction, so $HALT_{TM}$ is undecidable.

Undecidability “Direction”

<u>Fact</u>: Magic Wands Cannot Exist

Undecidability "Direction"

Fact: **Magic Wands Cannot Exist**

Use?	Hypothetical SuperSaw 2.0
Do?	Build a magic wand
Conclude?	**?**

Undecidability “Direction”

Fact: **Magic Wands Cannot Exist**

Use?	Hypothetical SuperSaw 2.0
Do?	Build a magic wand
Conclude?	SuperSaw 2.0 saw cannot exist

Undecidability "Direction"

Fact: **Magic Wands Cannot Exist**

Use?	**Something Assumed to Exist** Hypothetical SuperSaw 2.0
Do?	**Something Impossible** Build a magic wand
Conclude?	**Assumed Thing Cannot Exist** SuperSaw 2.0 saw cannot exist

Undecidability "Direction"

Fact: **Magic Wands Cannot Exist**

Use?	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	**Something Impossible** Build an A_{TM} Decider
Conclude?	**Assumed Thing Cannot Exist** **?**

Undecidability "Direction"

Fact: Magic Wands Cannot Exist

Use?	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	**Something Impossible** Build an A_{TM} Decider
Conclude?	**Assumed Thing Cannot Exist** $HALT_{TM}$ Decider can't exist

Undecidability "Direction"

<u>Fact</u>: Magic Wands Cannot Exist

Use?	Magic Wand	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	Bake a Cake	**Something Impossible** Build an A_{TM} Decider
Conclude?	**?**	**Assumed Thing Cannot Exist** $HALT_{TM}$ Decider can't exist

Undecidability "Direction"

Fact: Magic Wands Cannot Exist

Use?	Magic Wand	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	Bake a Cake	**Something Impossible** Build an A_{TM} Decider
Conclude?	**Nothing**	**Assumed Thing Cannot Exist** $HALT_{TM}$ Decider can't exist

Undecidability "Direction"

Fact: Magic Wands Cannot Exist

Use?	Magic Wand	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	Go Back in Time	**Something Impossible** Build an A_{TM} Decider
Conclude?		**Assumed Thing Cannot Exist** $HALT_{TM}$ Decider can't exist

Undecidability "Direction"

Fact: Magic Wands Cannot Exist

Use?	Magic Wand	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	Go Back in Time	**Something Impossible** Build an A_{TM} Decider
Conclude?	**Nothing**	**Assumed Thing Cannot Exist** $HALT_{TM}$ Decider can't exist

Undecidability "Direction"

Fact: **Magic Wands Cannot Exist**

Use?	**Something Impossible** Magic Wand	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	**Any Task** Go Back in Time	**Something Impossible** Build an A_{TM} Decider
Conclude?		**Assumed Thing Cannot Exist** $HALT_{TM}$ Decider can't exist

Undecidability "Direction"

Fact: **Magic Wands Cannot Exist**

Use?	**Something Impossible** Magic Wand	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	**Any Task** Go Back in Time	**Something Impossible** Build an A_{TM} Decider
Conclude?	**Nothing**	**Assumed Thing Cannot Exist** $HALT_{TM}$ Decider can't exist

Undecidability "Direction"

Fact: **Magic Wands Cannot Exist**

Use?	**Something Impossible** A_{TM} Decider	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	**Any Task** Build $HALT_{TM}$ Decider	**Something Impossible** Build an A_{TM} Decider
Conclude?		**Assumed Thing Cannot Exist** $HALT_{TM}$ Decider can't exist

Undecidability "Direction"

Fact: **Magic Wands Cannot Exist**

Use?	**Something Impossible** A_{TM} Decider	**Something Assumed to Exist** $HALT_{TM}$ Decider
Do?	**Any Task** Build $HALT_{TM}$ Decider	**Something Impossible** Build an A_{TM} Decider
Conclude?	**Nothing**	**Assumed Thing Cannot Exist** $HALT_{TM}$ Decider can't exist

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