NP-Complete
CSCI 338
Complexity Classes

Stuff we can solve in polynomial time.

Stuff we can verify solutions to in polynomial time.

\( P \)

\( NP \)
Complexity Classes

Stuff we can solve in polynomial time.

Stuff we can verify solutions to in polynomial time.

$\mathbb{NP}$-Complete

Stuff that can be used to solve everything in $\mathbb{NP}$ in polynomial extra time.

“Hardest problems in $\mathbb{NP}$”
$B$ is in $NP$-Complete if it satisfies two conditions:

1. $B \in NP$.
2. For every $A \in NP$, $A \leq_P B$. 

$B$ is in $NP$-Complete if it satisfies two conditions:

1. $B \in NP$.
2. For some $A \in NP-C$, $A \leq_P B$. 

$SAT$
NP-Complete

How to show something \((B)\) is in NP-Complete?

\(B\) is in NP-Complete if it satisfies two conditions:

1. \(B \in NP\).
2. For some \(A \in NP-C\), \(A \leq_p B\).
**NP-Complete**

How to show something \((B)\) is in \(NP\)-Complete?

1. Show it is in \(NP\).

\[ B \text{ is in } NP\text{-Complete if it satisfies two conditions:} \]
\[ 1. B \in NP. \]
\[ 2. \text{For some } A \in NP\text{-C, } A \leq_p B. \]
**NP-Complete**

How to show something \((B)\) is in \(NP\)-Complete?

1. Show it is in \(NP\).
2. Pick some known \(NP\)-Complete problem \(A\).

\[
B \text{ is in } NP\text{-Complete if it satisfies two conditions:} \\
1. B \in NP. \\
2. For some } A \in NP\text{-C, } A \leq_p B.
\]
NP-Complete

How to show something \((B)\) is in \(NP\)-Complete?

1. Show it is in \(NP\).
2. Pick some known \(NP\)-Complete problem \(A\).
3. Show that a solver for \(B\) can solve \(A\) in polynomial extra time.

\[ B \text{ is in } NP\text{-Complete if it satisfies two conditions:} \]
\[ 1. B \in NP. \]
\[ 2. \text{For some } A \in NP\text{-C, } \]
\[ A \leq_p B. \]
3SAT

Claim: 3SAT is in \( NP \)-Complete.

Proof:
3SAT

Claim: 3SAT is in NP-Complete.

Proof:

3SAT = \{⟨φ⟩: φ is a satisfiable formula with 3 variables per clause\}

\[ \phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2) \]

\[ (F \lor F \lor T) \land (T \lor F \lor F) \land (T \lor T \lor T) \]

\[
\begin{align*}
x_1 &= F & T \\
x_2 &= T \\
F \lor F \lor T &= T \\
T \lor F \lor F &= T \\
T \lor T \lor T &= T
\end{align*}
\]
3SAT

Claim: 3SAT is in \(NP\)-Complete.

Proof:

\(B\) is in \(NP\)-Complete if it satisfies two conditions:

1. \(B \in NP\).
2. For some \(A \in NP\)-C, \(A \leq_{P} B\).
3SAT

Claim: 3SAT is in NP-Complete.

Proof:

1. Show 3SAT is in NP.

B is in NP-Complete if it satisfies two conditions:
   1. $B \in NP$.
   2. For some $A \in NP-C$, $A \leq_p B$. 
3SAT

Claim: 3SAT is in $NP$-Complete.

Proof:

1. Show 3SAT is in $NP$.

   Given the Boolean formula and variable assignments, evaluate the formula and accept if true and reject if false. This can be done in $O(n)$ time where $n$ is the number of clauses.

$B$ is in $NP$-Complete if it satisfies two conditions:

1. $B \in NP$.
2. For some $A \in NP-C$, $A \leq_p B$. 
3SAT

Claim: 3SAT is in NP-Complete.

Proof:

1. Show 3SAT is in NP.

2. Show some NP-C problem can be solved using an algorithm for 3SAT.

$B$ is in NP-Complete if it satisfies two conditions:

1. $B \in NP$.
2. For some $A \in NP-C$, $A \leq_p B$. 
3SAT

Claim: 3SAT is in \( NP \)-Complete.

Proof:

1. Show 3SAT is in \( NP \).

2. Show some \( NP \)-C problem can be solved using an algorithm for 3SAT.

\( B \) is in \( NP \)-Complete if it satisfies two conditions:

1. \( B \in NP \).
2. For some \( A \in NP-C \), \( A \leq_p B \).
3SAT

Claim: 3SAT is in NP-Complete.

Proof:

1. Show 3SAT is in NP.

2. Show some NP-C problem can be solved using an algorithm for 3SAT.

B is in NP-Complete if it satisfies two conditions:

1. \( B \in NP \).
2. For some \( A \in NP-C \), \( A \leq_p B \).

Basic approach:

- Assume we have an algorithm that solves 3SAT.
- Use that algorithm to solve any instance to SAT.
$SAT \leq_p 3SAT$

Claim: $SAT \leq_p 3SAT$

Proof:
**SAT \leq_p 3SAT**

Claim: \( SAT \leq_p 3SAT \)

Proof:

Apollo 13 Filter Problem:

“We need to fit this into the hole for this, using nothing but that”
**SAT \leq_p 3SAT**

Claim: \( SAT \leq_p 3SAT \)

Proof:

Apollo 13 Filter Problem:

“We need to fit this into the hole for this, using nothing but that”

SAT Input \quad 3SAT Solver \quad Polynomial Time
Claim: $SAT \leq_p 3SAT$

Proof:
We need to turn instances of $SAT$ into instances of $3SAT$.

So we can use our $3SAT$ solver.
Claim: $SAT \leq_p 3SAT$

Proof:
We need to turn instances of $SAT$ into instances of $3SAT$.

So we can use our $3SAT$ solver.
Claim: $\text{SAT} \leq_p 3\text{SAT}$

Proof:
We need to turn instances of $\text{SAT}$ into instances of $3\text{SAT}$.

What is keeping our $\text{SAT}$ instance from being a $3\text{SAT}$ instance?

\[
\phi = (x_1) \land (\overline{x_1} \lor x_2 \lor x_2 \lor x_1) \land (\overline{x_1} \lor x_2) \\
\text{vs} \\
\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor x_2 \lor x_2) \land (\overline{x_1} \lor x_2 \lor x_2)
\]
Claim: $SAT \leq_p 3SAT$

Proof:
We need to turn instances of $SAT$ into instances of $3SAT$. If a clause has one literal?
\[ \text{SAT} \leq_p 3\text{SAT} \]

Claim: \( \text{SAT} \leq_p 3\text{SAT} \)

Proof:

We need to turn instances of \( \text{SAT} \) into instances of \( 3\text{SAT} \).

If a clause has one literal? \((x_1) \rightarrow (x_1 \lor x_1 \lor x_1)\)

If a clause has two literals?

\[ \text{SAT Solver} \]

\[ \text{SAT Input} \rightarrow 3\text{SAT Input} \rightarrow 3\text{SAT Solver} \rightarrow 3\text{SAT Solution} \rightarrow \text{SAT Solution} \]
**SAT \leq_p 3SAT**

Claim: SAT \leq_p 3SAT

Proof:
We need to turn instances of SAT into instances of 3SAT.
If a clause has one literal? \((x_1) \rightarrow (x_1 \lor x_1 \lor x_1)\)
If a clause has two literals? \((x_1 \lor x_2) \rightarrow (x_1 \lor x_1 \lor x_2)\)
If a clause had three literals?

[Diagram showing a flow from SAT Input through 3SAT Input to 3SAT Solver to 3SAT Solution to SAT Solution]
$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof:

We need to turn instances of $SAT$ into instances of $3SAT$.
If a clause has one literal? $(x_1) \rightarrow (x_1 \lor x_1 \lor x_1)$
If a clause has two literals? $(x_1 \lor x_2) \rightarrow (x_1 \lor x_1 \lor x_2)$
If a clause had three literals? No change.
If a clause has more than three literals?
Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.
$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

$\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)$
Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)$$

$$\rightarrow \phi_{3SAT} = \ ?$$
SAT $\leq_p$ 3SAT

Claim: SAT $\leq_p$ 3SAT

Proof: Convert SAT clauses with $> 3$ literals into 3SAT clauses.

$$\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)$$

$$\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)$$

\[\downarrow\]

**SAT Solver**

\[\begin{array}{c}
\text{SAT Input} \\
\xrightarrow{\text{3SAT Input}} \\
\xrightarrow{\text{3SAT Solver}} \\
\xrightarrow{\text{3SAT Solution}} \\
\text{SAT Solution}
\end{array}\]
Claim: \( SAT \leq_p 3SAT \)

Proof: Convert \( SAT \) clauses with > 3 literals into \( 3SAT \) clauses.

\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]

\[
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor \overline{z_1}) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)
\]
Claim: \( SAT \leq_p 3SAT \)

Proof: Convert \( SAT \) clauses with \( > 3 \) literals into \( 3SAT \) clauses.

\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]
\[
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z}_1 \lor x_3 \lor z_2) \land \cdots \land (\overline{z}_{k-3} \lor x_{k-1} \lor x_k)
\]

Need to show: \( \phi_{SAT} \) can be true \( \iff \phi_{3SAT} \) can be true.
**SAT \leq_P 3SAT**

Claim: \( SAT \leq_P 3SAT \)

Proof: Convert \( SAT \) clauses with \( > 3 \) literals into \( 3SAT \) clauses.

\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]

\[
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)
\]

Need to show: \( \phi_{SAT} \) can be true \( \iff \) \( \phi_{3SAT} \) can be true.

**SAT Solver**

Why if any only if? If \( \phi_{SAT} \) is satisfiable, we need to conclude that.
Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

$\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)$

$\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)$

Need to show: $\phi_{SAT}$ can be true $\iff \phi_{3SAT}$ can be true.

Why if any only if? If $\phi_{SAT}$ is satisfiable, we need to conclude that.
**SAT** $\leq_p$ **3SAT**

Claim: $\text{SAT} \leq_p \text{3SAT}$

Proof: Convert $\text{SAT}$ clauses with $> 3$ literals into $\text{3SAT}$ clauses.

$$\phi_{\text{SAT}} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)$$

$$\rightarrow \phi_{\text{3SAT}} = (x_1 \lor x_2 \lor z_1) \land (\overline{z}_1 \lor x_3 \lor z_2) \land \cdots \land (\overline{z}_{k-3} \lor x_{k-1} \lor x_k)$$

Need to show: $\phi_{\text{SAT}}$ can be true $\iff$ $\phi_{\text{3SAT}}$ can be true.

Why if any only if? If $\phi_{\text{SAT}}$ is satisfiable, we need to conclude that.

If $\phi_{\text{SAT}}$ is not satisfiable, we cannot conclude it is.
**SAT \leq_p 3SAT**

Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]

\[
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)
\]

Need to show: $\phi_{SAT}$ can be true $\iff$ $\phi_{3SAT}$ can be true.
**SAT \leq_p 3SAT**

Claim: \( SAT \leq_p 3SAT \)

Proof: Convert \( SAT \) clauses with > 3 literals into \( 3SAT \) clauses.

\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]

\[
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)
\]

Need to show: \( \phi_{SAT} \) can be true \( \iff \phi_{3SAT} \) can be true.

Suppose \( \phi_{SAT} \) can be true.
Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

\[ \phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k) \]
\[ \rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z}_1 \lor x_3 \lor z_2) \land \cdots \land (\overline{z}_{k-3} \lor x_{k-1} \lor x_k) \]

Need to show: $\phi_{SAT}$ can be true $\iff$ $\phi_{3SAT}$ can be true.

Suppose $\phi_{SAT}$ can be true. Then some $x_m$ is true.
**SAT \leq_p 3SAT**

Claim: SAT \leq_p 3SAT

Proof: Convert SAT clauses with > 3 literals into 3SAT clauses.

\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]
\[
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z}_1 \lor x_3 \lor z_2) \land \cdots \land (\overline{z}_{k-3} \lor x_{k-1} \lor x_k)
\]

Need to show: \(\phi_{SAT}\) can be true \(\iff\) \(\phi_{3SAT}\) can be true.

Suppose \(\phi_{SAT}\) can be true. Then some \(x_m\) is true.

\[
\ldots \land (\overline{z}_{i-1} \lor x_{m-1} \lor z_i) \land (\overline{z}_i \lor x_m \lor z_{i+1}) \land (\overline{z}_{i+1} \lor x_{m+1} \lor z_{i+2}) \land \ldots
\]
\textbf{SAT} \leq_p 3\text{SAT}

Claim: SAT \leq_p 3SAT

Proof: Convert SAT clauses with > 3 literals into 3SAT clauses.

\[ \phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k) \]
\[ \rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k) \]

Need to show: \( \phi_{SAT} \) can be true \( \iff \) \( \phi_{3SAT} \) can be true.

Suppose \( \phi_{SAT} \) can be true. Then some \( x_m \) is true. Let \( x_m \) be true in \( \phi_{3SAT} \).

\[ \cdots \land (\overline{z_{i-1}} \lor x_{m-1} \lor z_i) \land (\overline{z_i} \lor x_m \lor z_{i+1}) \land (\overline{z_{i+1}} \lor x_{m+1} \lor z_{i+2}) \land \cdots \]
**SAT \leq_p 3SAT**

Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

$\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)$

$\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)$

Need to show: $\phi_{SAT}$ can be true $\iff \phi_{3SAT}$ can be true.

Suppose $\phi_{SAT}$ can be true. Then some $x_m$ is true. Let $x_m$ be true in $\phi_{3SAT}$. Let all $z_i$’s before $x_m$ be true...

... $\land (\overline{z_{i-1}} \lor x_{m-1} \lor \text{ tagging in } z_i) \land \overline{z_i} \lor x_m \lor z_{i+1}) \land (\overline{z_{i+1}} \lor x_{m+1} \lor z_{i+2}) \land \ldots$
Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

$$
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)
$$

Need to show: $\phi_{SAT}$ can be true $\iff$ $\phi_{3SAT}$ can be true.

Suppose $\phi_{SAT}$ can be true. Then some $x_m$ is true. Let $x_m$ be true in $\phi_{3SAT}$. Let all $z_i$’s before $x_m$ be true and all $z_i$’s after be false.

$$
\ldots \land (\overline{z_{i-1}} \lor x_{m-1} \lor z_i) \land (\overline{z_i} \lor x_m \lor \overline{z_{i+1}}) \land (\overline{z_{i+1}} \lor x_{m+1} \lor z_{i+2}) \land \ldots
$$
SAT $\leq_P$ 3SAT

Claim: $SAT \leq_P 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

\[ \phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k) \]
\[ \rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z}_1 \lor x_3 \lor z_2) \land \cdots \land (\overline{z}_{k-3} \lor x_{k-1} \lor x_k) \]

Need to show: $\phi_{SAT}$ can be true $\iff$ $\phi_{3SAT}$ can be true.

Suppose $\phi_{SAT}$ can be true. Then some $x_m$ is true. Let $x_m$ be true in $\phi_{3SAT}$. Let all $z_i$’s before $x_m$ be true and all $z_i$’s after be false.

\[ \cdots \land (\overline{z}_{i-1} \lor x_{m-1} \lor z_i) \land (\overline{z}_i \lor x_m \lor \overline{z}_{i+1}) \land (\overline{z}_{i+1} \lor x_{m+1} \lor z_{i+2}) \land \cdots \]
Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

\[ \phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k) \]
\[ \Rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k) \]

Need to show: $\phi_{SAT}$ can be true $\iff$ $\phi_{3SAT}$ can be true.

Suppose $\phi_{SAT}$ can be true. Then some $x_m$ is true. Let $x_m$ be true in $\phi_{3SAT}$. Let all $z_i$'s before $x_m$ be true and all $z_i$'s after be false.

$\Rightarrow$ Every clause has a variable set to true.
\[ SAT \leq_p 3SAT \]

Claim: \( SAT \leq_p 3SAT \)

Proof: Convert \( SAT \) clauses with \( > 3 \) literals into \( 3SAT \) clauses.

\[ \phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k) \]
\[ \rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z}_1 \lor x_3 \lor z_2) \land \cdots \land (\overline{z}_{k-3} \lor x_{k-1} \lor x_k) \]

Need to show: \( \phi_{SAT} \) can be true \( \iff \phi_{3SAT} \) can be true.

Suppose \( \phi_{SAT} \) can be true. Then some \( x_m \) is true. Let \( x_m \) be true in \( \phi_{3SAT} \). Let all \( z_i \)'s before \( x_m \) be true and all \( z_i \)'s after be false.

\[ \Rightarrow \] Every clause has a variable set to true. \( \therefore \phi_{3SAT} = T. \)
SAT \leq_p 3SAT

Claim: SAT \leq_p 3SAT

Proof: Convert SAT clauses with > 3 literals into 3SAT clauses.

\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]

\[
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)
\]

Need to show: \(\phi_{SAT}\) can be true \(\iff\) \(\phi_{3SAT}\) can be true.

Suppose \(\phi_{SAT}\) can be true. Then some \(x_m\) is true. Let \(x_m\) be true in \(\phi_{3SAT}\). Let all \(z_i\)'s before \(x_m\) be true and all \(z_i\)'s after be false.

\(\Rightarrow\) Every clause has a variable set to true. \(\therefore\) \(\phi_{3SAT} = T\).

Suppose \(\phi_{3SAT}\) can be true.
\textbf{SAT} \leq_p 3\textit{SAT}

Claim: \( SAT \leq_p 3SAT \)

Proof: Convert \( SAT \) clauses with \( > 3 \) literals into \( 3SAT \) clauses.
\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]
\[
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)
\]

Need to show: \( \phi_{SAT} \) can be true \( \iff \phi_{3SAT} \) can be true.

Suppose \( \phi_{SAT} \) can be true. Then some \( x_m \) is true. Let \( x_m \) be true in \( \phi_{3SAT} \). Let all \( z_i \)'s before \( x_m \) be true and all \( z_i \)'s after be false.
\[
\Rightarrow \quad \text{Every clause has a variable set to true.} \quad \therefore \phi_{3SAT} = T.
\]

Suppose \( \phi_{3SAT} \) can be true. Some \( x_m \) must be true.
Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]

\[
\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)
\]

Need to show: $\phi_{SAT}$ can be true $\iff$ $\phi_{3SAT}$ can be true.

Suppose $\phi_{SAT}$ can be true. Then some $x_m$ is true. Let $x_m$ be true in $\phi_{3SAT}$. Let all $z_i$’s before $x_m$ be true and all $z_i$’s after be false.

$\Rightarrow$ Every clause has a variable set to true. $\therefore \phi_{3SAT} = T$.

Suppose $\phi_{3SAT}$ can be true. Some $x_m$ must be true. If not, all $z_i$’s must be true, and last clause would be false.
**SAT \leq_p 3SAT**

Claim: SAT \leq_p 3SAT

Proof: Convert SAT clauses with > 3 literals into 3SAT clauses.

\[
\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)
\]
\[
\Rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)
\]

Need to show: \(\phi_{SAT}\) can be true \iff \(\phi_{3SAT}\) can be true.

Suppose \(\phi_{SAT}\) can be true. Then some \(x_m\) is true. Let \(x_m\) be true in \(\phi_{3SAT}\). Let all \(z_i\)'s before \(x_m\) be true and all \(z_i\)'s after be false.
\[
\Rightarrow \text{Every clause has a variable set to true.} \ \therefore \ \phi_{3SAT} = T.
\]

Suppose \(\phi_{3SAT}\) can be true. Some \(x_m\) must be true. If not, all \(z_i\)'s must be true, and last clause would be false. \(\therefore \phi_{SAT} = T\).
**$SAT \leq_p 3SAT$**

Claim: $SAT \leq_p 3SAT$

Proof: Convert $SAT$ clauses with $> 3$ literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)$$

$$\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor z_1) \land (\overline{z_1} \lor x_3 \lor z_2) \land \cdots \land (\overline{z_{k-3}} \lor x_{k-1} \lor x_k)$$

Need to show: $\phi_{SAT}$ can be true $\iff$ $\phi_{3SAT}$ can be true.

Suppose $\phi_{SAT}$ can be true. Then some $x_m$ is true. Let $x_m$ be true in $\phi_{3SAT}$. Let all $z_i$’s before $x_m$ be true and all $z_i$’s after be false.

$\Rightarrow$ Every clause has a variable set to true. $\therefore \phi_{3SAT} = T$

Suppose $\phi_{3SAT}$ can be true. Some $x_m$ must be true. If not, all $z_i$’s must be true, and last clause would be false. $\therefore \phi_{SAT} = T$

$\therefore SAT \leq_p 3SAT$
3SAT

Claim: 3SAT is in NP-Complete.

Proof:

1. 3SAT is in NP. ✓

2. SAT $\leq_p$ 3SAT ✓

Therefore, 3SAT is in NP-Complete.

\[
\begin{align*}
B \text{ is in } NP-\text{Complete if it satisfies two conditions:} \\
1. & \ B \in NP. \\
2. & \text{For some } A \in NP-C, A \leq_p B.
\end{align*}
\]
$NP - C$

Cook-Levin Theorem

All of NP

"Can be solved by"

SAT
$NP - C$

All of NP

Cook-Levin Theorem

"Can be solved by"

SAT

Today’s Class

3SAT
CLIQUE

Clique: a subgraph where every pair of nodes share an edge (i.e. a complete subgraph).

$k$-Clique: A clique that contains $k$ vertices.

\[ CLIQUE = \{ (G,k) : G \text{ is an undirected graph with a } k\text{-clique} \} \]
Claim: \textit{CLIQUE} $\in$ \textit{NP}-Complete

Proof:
Claim: $\text{CLIQUE} \in \text{NP-Complete}$

Proof:

1. ???
Claim: $CLIQUE \in NP$-Complete

Proof:

1. $CLIQUE \in NP$
Claim: \textit{CLIQUE} $\in \text{NP}$-Complete

Proof:

1. \textit{CLIQUE} $\in \text{NP}$

Given a graph $G = (V, E)$, where $|V| = n$, and a subset $S \subseteq V$, where $|S| \geq k$, check if all pairs of vertices in $S$ are in $E$. Running time: $O(n^2)$. 
**CLIQUE**

Claim: \( \text{CLIQUE} \in \text{NP-Complete} \)

Proof:

1. \( \text{CLIQUE} \in \text{NP} \)

   Given a graph \( G = (V, E) \), where \( |V| = n \), and a subset \( S \subseteq V \), where \( |S| \geq k \), check if all pairs of vertices in \( S \) are in \( E \). Running time: \( O(n^2) \).

2. ???
Claim: $CLIQUE \in NP$-Complete

Proof:

1. $CLIQUE \in NP$

Given a graph $G = (V, E)$, where $|V| = n$, and a subset $S \subseteq V$, where $|S| \geq k$, check if all pairs of vertices in $S$ are in $E$. Running time: $O(n^2)$.

2. $3SAT \leq_P CLIQUE$
Claim: \(3SAT \leq_p CLIQUE\)

Proof:
Claim: $3SAT \leq_p CLIQUE$

Proof:

$3SAT$ reduces to $CLIQUE$ if $3SAT$ can be solved with a solver for $CLIQUE$. 
Claim: $3SAT \leq_p CLIQUE$

Proof:

Apollo 13 Filter Problem:

“We need to fit this into the hole for this, using nothing but that”
Claim: $3SAT \leq_p CLIQUE$

Proof:

\[ \phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2) \land (x_1 \lor x_2 \lor x_2) \]
Claim: $3SAT \leq_p CLIQUE$

Proof:

\[ \phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (x_1 \lor x_2 \lor x_2) \]

\[ \phi - \text{Satisfiable} \iff \exists k - \text{Clique} \]
**CLIQUE**

Claim: $3SAT \leq_p CLIQUE$

Proof: Let $\phi$ be a formula with $k$ clauses. Generate an undirected graph $G$:

$$\phi = (x_1 \lor \overline{x_1} \lor x_2) \land$$
$$\overline{x_1} \lor x_2 \lor \overline{x_2} \land$$
$$\overline{x_1} \lor \overline{x_2} \lor x_2 \land$$
$$\overline{x_1} \lor x_2 \lor x_2$$
**CLIQUE**

Claim: $3SAT \leq_p CLIQUE$

Proof: Let $\phi$ be a formula with $k$ clauses. Generate an undirected graph $G$:
For each clause in $\phi$, make a node for each literal.

\[
\phi = (x_1 \lor x_1 \lor x_2) \land \\
(x_1 \lor \overline{x_2} \lor \overline{x_2}) \land \\
(x_1 \lor x_2 \lor x_2)
\]