CLIQUE
CSCI 338
Project 3

Independent Set: Given a graph $G = (V, E)$ and integer $k \leq |V|$, is there a subset $V'$ of size $\geq k$, such that no two vertices $\in V'$ are adjacent?
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What performance metrics do we care about?
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Accuracy, speed.
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Size of VC (IS) found

Running time
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“Size” of graph (# vertices, # edges, connectivity)
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One test on a graph of 30 vertices?
No! Average of X iterations.

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Size of VC (IS) found

One test on a graph of 30 vertices?
No! Average of X iterations.

Running time

Exact result on random graph of 30 vertices and inexact result on different random graph of 30 vertices?
No! results on same random graph.

“Size” of graph (# vertices, # edges, connectivity)
$NP$-Complete

$B$ is in $NP$-Complete if it satisfies two conditions:
1. $B \in NP$.
2. For some $A \in NP$-C, $A \leq_P B$.

How to show something ($B$) is in $NP$-Complete?
1. Show it is in $NP$.
2. Pick some known $NP$-Complete problem $A$.
3. Show that a solver for $B$ can solve $A$ in polynomial extra time.
**CLIQUE**

**Clique**: a subgraph where every pair of nodes share an edge (i.e. a complete subgraph).

**$k$-Clique**: A clique that contains $k$ vertices.

$CLIQUE = \{ (G, k) : G \text{ is an undirected graph with a } k\text{-clique} \}$
Claim: \textit{CLIQUE} $\in NP$-Complete

Proof:
Claim: $CLIQUE \in NP$-Complete

Proof:

1. ???
**CLIQUE**

Claim: $CLIQUE \in NP$-Complete

Proof:

1. $CLIQUE \in NP$
**CLAIM**

Claim: $\text{CLIQUE} \in NP$-Complete

Proof:

1. $\text{CLIQUE} \in NP$

   Given a graph $G = (V, E)$, where $|V| = n$, and a subset $S \subseteq V$, where $|S| \geq k$, check if all pairs of vertices in $S$ are in $E$. Running time: $O(n^2)$. 
Claim: $\text{CLIQUE} \in \text{NP}$-Complete

Proof:

1. $\text{CLIQUE} \in \text{NP}$
   
   Given a graph $G = (V, E)$, where $|V| = n$, and a subset $S \subseteq V$, where $|S| \geq k$, check if all pairs of vertices in $S$ are in $E$. Running time: $O(n^2)$.

2. ???
**CLIQUE**

Claim: \textit{CLIQUE} $\in \text{NP}$-Complete

Proof:

1. \textit{CLIQUE} $\in \text{NP}$

   Given a graph $G = (V, E)$, where $|V| = n$, and a subset $S \subseteq V$, where $|S| \geq k$, check if all pairs of vertices in $S$ are in $E$. Running time: $O(n^2)$.

2. \text{3SAT} \leq_p \text{CLIQUE}
Claim: \(3SAT \leq_p CLIQUE\)

Proof:
Claim: $3SAT \leq_p CLIQUE$

Proof:

$3SAT$ reduces to $CLIQUE$ if $3SAT$ can be solved with a solver for $CLIQUE$. 
**CLIQUE**

Claim: $3SAT \leq_p CLIQUE$

Proof:  

Apollo 13 Filter Problem:  
"We need to fit this into the hole for this, using nothing but that"
Claim: $3SAT \leq_p CLIQUE$

Proof:

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$$
CLAIM

Claim: $3SAT \leq_p CLIQUE$

Proof:

$$\phi = (x_1 \lor x_1 \lor x_2) \land \overline{(x_1 \lor x_2 \lor x_2)} \land (x_1 \lor x_2 \lor x_2)$$

$\phi$ - Satisfiable $\iff \exists k$ - Clique
Claim: $3SAT \leq_p CLIQUE$

Proof: Let $\phi$ be a formula with $k$ clauses. Generate an undirected graph $G$:

\[
\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)
\]
Claim: $3SAT \leq_p CLIQUE$

Proof: Let $\phi$ be a formula with $k$ clauses. Generate an undirected graph $G$:

For each clause in $\phi$, make a node for each literal.

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$$
CLIQUE

Claim: $3SAT \leq_P CLIQUE$

Proof: Let $\phi$ be a formula with $k$ clauses. Generate an undirected graph $G$:

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.

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**Polynomial Time?**

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$$
Claim: $3SAT \leq_p CLIQUE$

Proof: Let $\phi$ be a formula with $k$ clauses. Generate an undirected graph $G$:
For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:
1. Nodes in the same clause
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Polynomial Time?
Yes
- $3k$ nodes
- $O(k^2)$ edges (fewer than complete graph)
**CLIQUE**

Claim: $3SAT \leq_p CLIQUE$

Proof: Let $\phi$ be a formula with $k$ clauses. Generate an undirected graph $G$:
For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.

Need to show: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor x_2 \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor \overline{x_2})$$
**CLAIM**

Claim: $3SAT \leq_p CLIQUE$

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Need to show: $\phi$ is satisfiable $\iff G$ has a $k$-clique.
**CLIQUE**

Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:
1. Nodes in the same clause
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**CLAIM**

Claim: \(3SAT \leq_P CLIQUE\)

Proof: \(\phi\) is satisfiable \(\iff G\) has a \(k\)-clique.

⇒ Suppose \(\phi\) is satisfiable. Then...

\[
\phi = (x_1 \lor x_1 \lor x_2) \land \\
    (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land \\
    (\overline{x_1} \lor x_2 \lor x_2)
\]

For each clause in \(\phi\), make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause.

$$\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor x_2 \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$$

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals.

$$\phi = (x_1 \lor \overline{x_1} \lor x_2) \land 
\overline{(x_1 \lor x_2 \lor \overline{x_2})} \land 
\overline{(x_1 \lor x_2 \lor \overline{x_2})}$$

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: 3SAT \leq_p CLIQUE

Proof: \phi is satisfiable \iff G has a k-clique.

\Rightarrow Suppose \phi is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in G for one of the true literals. This forms a \textbf{k-clique}, since...

\[
\phi = (x_1 \lor x_1 \lor x_2) \land \\
(x_1 \lor x_2 \lor x_2) \land \\
(x_1 \lor x_2 \lor x_2)
\]

For each clause in \phi, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since $k$ nodes are selected and each is joined by an edge.

$$
\phi = (x_1 \lor x_1 \lor x_2) \land \\
(\overline{x_1} \lor x_2 \lor \overline{x_2}) \land \\
(\overline{x_1} \lor x_2 \lor x_2)
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**CLIQUE**

Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

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$$\phi = (x_1 \lor x_1 \lor x_2) \land \overline{(x_1 \lor x_2 \lor x_2)} \land (\overline{x_1} \lor x_2 \lor x_2)$$

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\implies$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since $k$ nodes are selected and each is joined by an edge.

$\impliedby$ Suppose $G$ has a $k$-clique.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
**CLAIM**

Claim: $3SAT \leq_p CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since $k$ nodes are selected and each is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a node from the $k$-clique in each clause.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
**CLIQUE**

Claim: $3SAT \leq_P CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since $k$ nodes are selected and each is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a node from the $k$-clique in each clause. (nodes in the same clause can’t share an edge!)

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. **Nodes in the same clause**
2. **Nodes that are negations of each other.**
Claim: $3SAT \leq_P CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal $\psi = (x_1 \lor x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor x_2)$ is true in each clause. For each clause, select a node for one of the true literals. Since $k$ nodes are selected and each is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a node from the $k$-clique in each clause.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
**Claim:** $3SAT \leq_P CLIQUE$

**Proof:** $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal in each clause is true. For each clause $\phi = (x_1 \lor \overline{x_1} \lor x_2) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2 \lor \overline{x_2})$, there are at least $k$ distinct nodes that are true. Selecting $k$ such nodes forms a $k$-clique, since each node is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a node from the $k$-clique in each clause. Making each node in the $k$-clique true results in...

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
**CLIQUE**

Claim: $3SAT \leq_P CLIQUE$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal in $\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_2) \land (x_1 \lor x_2 \lor x_2)$ is true. For each clause, select a node for one of the true literals. Since $k$ nodes are selected and each is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a node from the $k$-clique in each clause. Making each node in the $k$-clique true results in $\phi$ being true.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
Claim: $3\text{SAT} \leq_p \text{CLIQUE}$

Proof: $\phi$ is satisfiable $\iff G$ has a $k$-clique.

$\Rightarrow$ Suppose $\phi$ is satisfiable. Then at least one literal is true in each clause. For each clause, select a node in $G$ for one of the true literals. This forms a $k$-clique, since $k$ nodes are selected and each is joined by an edge.

$\Leftarrow$ Suppose $G$ has a $k$-clique. Then there is a node from the $k$-clique in each clause. Making each node in the $k$-clique true results in $\phi$ being true.

For each clause in $\phi$, make a node for each literal. Make edge between every pair of nodes, except:

1. Nodes in the same clause
2. Nodes that are negations of each other.
**CLIQUE**

Claim: $CLIQUE \in NP$-Complete

Proof:

1. $CLIQUE \in NP$

   Given a graph $G = (V, E)$, where $|V| = n$, and a subset $S \subseteq V$, where $|S| \geq k$, check if all pairs of vertices in $S$ are in $E$. Running time: $O(n^2)$.

2. $3SAT \leq_p CLIQUE$

\[\therefore CLIQUE \in NP - C\]
**CLIQUE**

Claim: \( \text{CLIQUE} \in \text{NP}-\text{Complete} \)

Proof:

1. \( \text{CLIQUE} \in \text{NP} \)

   Given a graph \( G = (V, E) \), where \( |V| = n \), and a subset \( S \subseteq V \), where \( |S| \geq k \), check if all pairs of vertices in \( S \) are in \( E \). Running time: \( O(n^2) \).

2. \( 3\text{SAT} \leq_p \text{CLIQUE} \)

   ![Diagram showing 3SAT Solver, CLIQUE Input, CLIQUE Solver, CLIQUE Solution, and 3SAT Solution]

\[ \therefore \text{CLIQUE} \in \text{NP}-\text{C} \]