Undecidability
CSCI 338
Question: Does the government have aliens?

Yes!

Yes, the government has aliens.

Bread Recipe:
1. Add flour, water, sugar, yeast, salt.
2. If the government has aliens, add an egg.
3. Mix, bake, eat.

Frank
Can always perfectly answer the question: “Does a recipe use eggs?”
Does the government have aliens?:
1. Write down the following recipe:

   Bread Recipe:
   1. Add flour, water, sugar, yeast, salt.
   2. If the government has alien, add an egg.
   3. Mix, bake, eat.

2. Ask Frank if the recipe uses eggs.
3. If Frank says yes, yes, they got aliens. If Frank says no, no, they don’t.
$E_{TM}$

Claim: $E_{TM} = \{\langle M \rangle: M \text{ is a TM and } L(M) = \emptyset \}$ is undecidable.

Proof: Suppose $E_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.
   2. Run $H$ on $\langle M_2 \rangle$.
   3. If $H$ accepts, reject. If $H$ rejects, accept.
Does the government have aliens?:
1. Write down the following recipe:
   
   Bread Recipe:
   1. Add flour, water, sugar, yeast, salt.
   2. If the government has alien, add an egg.
   3. Mix, bake, eat.
   
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3. If Frank says yes, yes, they got aliens. If Frank says no, no, they don’t.

$A_{TM}$ Decider (does $N$ accept $\omega$?):
1. Construct TM $M_2$ on input $\langle x \rangle$:
   
   1. Run $N$ on $\omega$ and accept if $N$ does.

   2. Run $H$ on $\langle M_2 \rangle$.

   3. If $H$ accepts, reject. If $H$ rejects, accept.
Question?

Does the government have aliens?:

1. Write down the following recipe:
   
   **Bread Recipe:**
   1. Add flour, water, sugar, yeast, salt.
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2. Ask Frank if the recipe uses eggs.
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\[ A_{TM} \text{ Decider (does } N \text{ accept } \omega?) : \]

1. Construct TM \( M_2 \) on input \( \langle x \rangle \):
   1. Run \( N \) on \( \omega \) and accept if \( N \) does.
2. Run \( H \) on \( \langle M_2 \rangle \).
3. If \( H \) accepts, reject. If \( H \) rejects, accept.
Does the government have aliens?:
1. Write down the following recipe:

   **Bread Recipe:**
   1. Add flour, water, sugar, yeast, salt.
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$A_{TM}$ Decider (does $N$ accept $\omega$?):
1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.

2. Run $H$ on $\langle M_2 \rangle$.
3. If $H$ accepts, reject. If $H$ rejects, accept.

Available Tool
Does the government have aliens?:

1. Write down the following recipe:
   - **Bread Recipe:**
     1. Add flour, water, sugar, yeast, salt.
     2. If the government has alien, add an egg.
     3. Mix, bake, eat.

2. Ask Frank if the recipe uses eggs.
3. If Frank says yes, **yes, they got aliens.** If Frank says no, **no, they don’t.**

Available Tool

Recipe uses egg $\iff$ Government has aliens

$L(M_2) \neq \emptyset \iff N$ accepts $\omega$

$A_{TM}$ Decider (does $N$ accept $\omega$?):

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.

2. Run $H$ on $\langle M_2 \rangle$.
3. If $H$ accepts, reject. If $H$ rejects, accept.

Recipe/algorithm/TM that forces the tool to answer the question.
Claim: $E_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) = \emptyset \}$ is undecidable.

Proof: Suppose $E_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{ on input } \langle N, \omega \rangle$

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TODO: Argue that $S$ is a decider for $A_{TM}$
Claim: $E_{TM} = \{\langle M \rangle: M \text{ is a TM and } L(M) = \emptyset \}$ is undecidable.

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   2. Run $H$ on $\langle M_2 \rangle$.
   3. If $H$ accepts, reject. If $H$ rejects, accept.

If $N$ accepts $\omega$, ... $S$ will accept. If $N$ does not accept $\omega$, ... $S$ will reject.
Claim: $E_{TM} = \{\langle M \rangle: M \text{ is a TM and } L(M) = \emptyset\}$ is undecidable.

Proof: Suppose $E_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

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2. Run $H$ on $\langle M_2 \rangle$.
3. If $H$ accepts, reject. If $H$ rejects, accept.

If $N$ accepts $\omega$, $L(M_2) = \Sigma^*$, ...
**Claim:** $E_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) = \emptyset \}$ is undecidable.

**Proof:** Suppose $E_{TM}$ is decidable and let TM $H$ be its decider.

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   1. Run $N$ on $\omega$ and accept if $N$ does.
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   3. If $H$ accepts, reject. If $H$ rejects, accept.

If $N$ accepts $\omega$, $L(M_2) = \Sigma^*$, $H$ will reject, ...
Claim: $E_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) = \emptyset \}$ is undecidable.

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   2. Run $H$ on $\langle M_2 \rangle$.
   3. If $H$ accepts, reject. If $H$ rejects, accept.

If $N$ accepts $\omega$, $L(M_2) = \Sigma^*$, $H$ will reject, and $S$ will accept.
Claim: $E_{TM} = \{ \langle M \rangle : M$ is a TM and $L(M) = \emptyset \}$ is undecidable.

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   1. Run $N$ on $\omega$ and accept if $N$ does.
   2. Run $H$ on $\langle M_2 \rangle$.
   3. If $H$ accepts, reject. If $H$ rejects, accept.

If $N$ accepts $\omega$, $L(M_2) = \Sigma^*$, $H$ will reject, and $S$ will accept. If $N$ does not accept $\omega$, $L(M_2) = \emptyset$, $H$ will accept, and $S$ will reject.
Claim: $E_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) = \emptyset \}$ is undecidable.

Proof: Suppose $E_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S =$ on input $\langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.
   2. Run $H$ on $\langle M_2 \rangle$.
   3. If $H$ accepts, reject. If $H$ rejects, accept.

If $N$ accepts $\omega$, $L(M_2) = \Sigma^*$, $H$ will reject, and $S$ will accept. If $N$ does not accept $\omega$, $L(M_2) = \emptyset$, $H$ will accept, and $S$ will reject. Thus, $S$ decides $A_{TM}$, which is a contradiction, so $E_{TM}$ is undecidable.
Claim: $EQ_{TM} = \{⟨A, B⟩: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof:
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. ???

To show $EQ_{TM}$ is undecidable, use it to decide $A_{TM}$. 
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}:
\begin{align*}
S &= \text{on input } \langle N, \omega \rangle \\
1. &\quad ???
\end{align*}

We have a way ($H$) to test if two TMs have the same language. How could we use that to test if a TM’s accepts some input?

Plan: ?
Claim: $EQ_{TM} = \{\langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. ???

We have a way ($H$) to test if two TMs have the same language.

How could we use that to test if a TM’s accepts some input?

Plan: Make two TMs that have the same language if and only if $N$ accepts $\omega$. 

\[ \]
$EQ_{TM}$

Claim: $EQ_{TM} = \{ \langle A, B \rangle : A, B$ are TMs and $L(A) = L(B) \}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_1$ on input $\langle x \rangle$:

   1. accept.
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_1$ on input $\langle x \rangle$:
   1. accept.
$EQ_{TM}$

Claim: $EQ_{TM} = \{\langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_1$ on input $\langle x \rangle$:
   1. accept.

$L(M_2) = \Sigma^*$
Claim: \( EQ_{TM} = \{ \langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \} \) is undecidable.

Proof: Suppose \( EQ_{TM} \) is decidable and let TM \( H \) be its decider.

Build a TM \( S \) that decides \( A_{TM} \):

\[ S = \text{on input } \langle N, \omega \rangle \]

1. Construct TM \( M_1 \) on input \( \langle x \rangle \):
   1. accept.
2. Construct TM \( M_2 \) on input \( \langle y \rangle \):
   1. ?
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_1$ on input $\langle x \rangle$:
   1. accept.

2. Construct TM $M_2$ on input $\langle y \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.
Claim: $EQ_{TM} = \{\langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

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1. Construct TM $M_1$ on input $\langle x \rangle$:
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Claim: $EQ_{TM} = \{\langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \}$ is undecidable.

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2. Construct TM $M_2$ on input $\langle y \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.

3. Run $H$ on $\langle M_1, M_2 \rangle$.
4. If $H$ accepts, ???

$N$ accepts $\omega$ \iff $L(M_2) = \Sigma^*$
Claim: $EQ_{TM} = \{ \langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \}$ is undecidable.

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   1. Run $N$ on $\omega$ and accept if $N$ does.

3. Run $H$ on $\langle M_1, M_2 \rangle$.

4. If $H$ accepts, accept. If $H$ rejects, reject.
Claim: \( EQ_{TM} = \{ \langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \} \) is undecidable.

Proof: Suppose \( EQ_{TM} \) is decidable and let TM \( H \) be its decider.

Build a TM \( S \) that decides \( A_{TM} \):

\[
S = \text{on input } \langle N, \omega \rangle \\
1. \text{Construct TM } M_1 \text{ on input } \langle x \rangle : \\
   1. \text{ accept.} \\
2. \text{Construct TM } M_2 \text{ on input } \langle y \rangle : \\
   1. \text{ Run } N \text{ on } \omega \text{ and accept if } N \text{ does.} \\
3. \text{ Run } H \text{ on } \langle M_1, M_2 \rangle. \\
4. \text{ If } H \text{ accepts, accept. If } H \text{ rejects, reject.}
\]

If \( N \) accepts \( \omega \), then \( M_1 \) and \( M_2 \) have the same language (\( \Sigma^* \)) and \( S \) accepts. If \( N \) does not accept \( \omega \), then they have different languages and \( S \) rejects. Thus, \( S \) decides \( A_{TM} \), which is a contradiction, so \( EQ_{TM} \) is undecidable.
Claim: $EQ_{TM} = \{\langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S =$ on input $\langle N, \omega \rangle$

1. Construct TM $M_1$ on input $\langle x \rangle$:
   1. accept.
2. Construct TM $M_2$ on input $\langle y \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.
3. Run $H$ on $\langle M_1, M_2 \rangle$.
4. If $H$ accepts, accept. If $H$ rejects, reject.

$N$ accepts $\omega$ \iff $L(M_2) = \Sigma^*$

Lots of ways to write the recipe!
Claim: $EQ_{TM} = \{ \langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

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   1. accept.

2. Construct TM $M_2$ on input $\langle y \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.

3. Run $H$ on $\langle M_1, M_2 \rangle$.

4. If $H$ accepts, accept. If $H$ rejects, reject.

$N$ accepts $\omega$ ⇔ $L(M_1) = L(M_2)$

Lots of ways to write the recipe!
Question: Does the government have aliens?

Bread Recipe:
1. Add flour, water, sugar, yeast, salt.
2. If government has aliens, add an egg.
3. Mix, bake, eat.
Question: Does the government have aliens?

Bread Recipe:
1. Add flour, water, sugar, yeast, salt.
2. If government doesn’t have aliens, add an egg.
3. Mix, bake, eat.

Recipe:
1. If government has aliens, add an egg.
Question: Does the government have aliens?

Bread Recipe:
1. Add flour, water, sugar, yeast, salt.
2. If government doesn’t have aliens, add an egg.
3. Mix, bake, eat.

Won’t Work Recipe:
1. If government has aliens, add flour.
2. Add eggs.

Recipe:
1. If government has aliens, add an egg.
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Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = $ on input $\langle N, \omega \rangle$

1. Construct TM $M_1$ on input $\langle x \rangle$:
   1. accept.
2. Construct TM $M_2$ on input $\langle y \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.
3. Run $H$ on $\langle M_1, M_2 \rangle$.
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Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_1$ on input $\langle x \rangle$:
   1. reject.
2. Construct TM $M_2$ on input $\langle y \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.
3. Run $H$ on $\langle M_1, M_2 \rangle$.
4. If $H$ accepts, reject. If $H$ rejects, accept.

Lots of ways to write the recipe!
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1. Construct TM $M_1$ on input $\langle x \rangle$:
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2. Construct TM $M_2$ on input $\langle y \rangle$:
   1. Run $N$ on $\omega$.
   2. If $N$ rejects, reject.
   3. accept.
3. Run $H$ on $\langle M_1, M_2 \rangle$.
4. If $H$ accepts, accept. If $H$ rejects, reject.