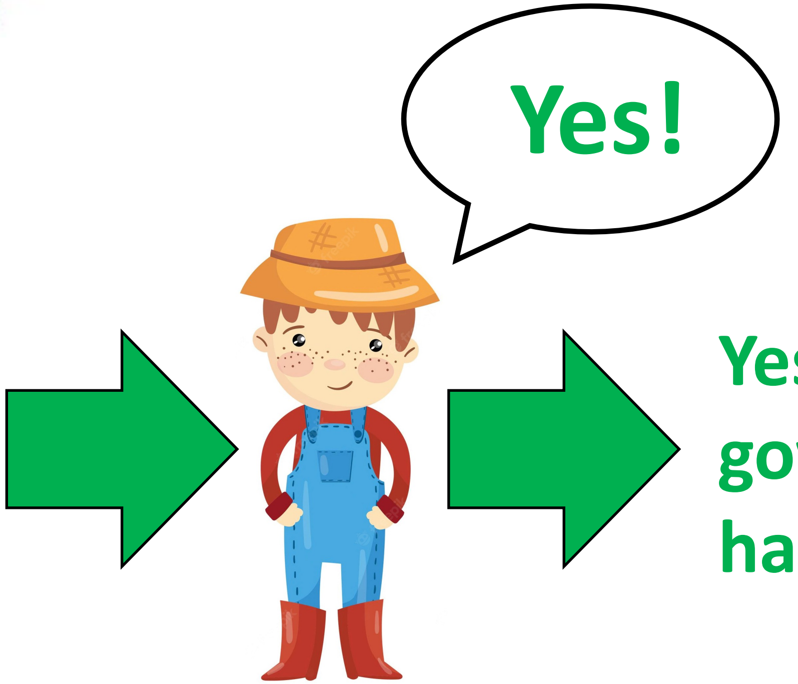
Undecidability

CSCI 338

Question: Does the government have aliens?

Bread Recipe:

1. Add flour, water, sugar, yeast, salt.
2. If the government has aliens, add an egg.
3. Mix, bake, eat.



Yes, the government has aliens.

Frank

Can always perfectly answer the question: "Does a recipe use eggs?"

Does the government have aliens?:

1. Write down the following recipe:

   Bread Recipe:
   1. Add flour, water, sugar, yeast, salt.
   2. If the government has alien, add an egg.
   3. Mix, bake, eat.

2. Ask Frank if the recipe uses eggs.
3. If Frank says yes, <u>yes, they got aliens</u>. If Frank says no, <u>no, they don't</u>.

E_{TM}

Claim: $E_{TM} = \{\langle M \rangle : M$ is a TM and $L(M) = \emptyset\}$ is undecidable.

Proof: Suppose E_{TM} is decidable and let TM H be its decider.

Build a TM S that decides A_{TM}:

S = on input $\langle N, \omega \rangle$

1. Construct TM M_2 on input $\langle x \rangle$:
   1. Run N on ω and <u>accept</u> if N does.
2. Run H on $\langle M_2 \rangle$.
3. If H accepts, <u>reject</u>. If H rejects, <u>accept</u>.

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A_{TM} Decider (does N accept ω?):

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Available Tool

Question?

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Available Tool

Recipe uses egg $\Leftrightarrow$ Government has aliens

$L(M_2) \neq \emptyset \Leftrightarrow N$ **accepts** ω

Recipe/algorithm/TM that forces the tool to answer the question.

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TODO: Argue that S is a decider for A_{TM}

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3. If H accepts, reject. If H rejects, accept.

If N accepts ω, … S will accept. If N does not accept ω, … S will reject.

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If N accepts ω, $L(M_2) = \Sigma^*$, …

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3. If H accepts, <u>reject</u>. If H rejects, <u>accept</u>.

If N accepts ω, $L(M_2) = \Sigma^*$, H will reject, and S will accept. If N does not accept ω, $L(M_2) = \emptyset$, H will accept, and S will reject. Thus, S decides A_{TM}, which is a contradiction, so E_{TM} is undecidable.

EQ_{TM}

Claim: $EQ_{TM} = \{\langle A, B\rangle : A, B$ are TMs and $L(A) = L(B)\}$ is undecidable.

Proof:

?

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1. ???

To show EQ_{TM} is undecidable, use it to decide A_{TM}.

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Proof: Suppose EQ_{TM} is decidable and let TM H be its decider.

Build a TM S that decides A_{TM}:

S = on input $\langle N, \omega\rangle$

1. ???

We have a way (H) to test if two TMs have the same language.

How could we use that to test if a TM's accepts some input?

Plan: ?

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Build a TM S that decides A_{TM}:

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1. ???

We have a way (H) to test if two TMs have the same language.

How could we use that to test if a TM's accepts some input?

Plan: Make two TMs that have the same language if and only if N accepts ω.

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Proof: Suppose EQ_{TM} is decidable and let TM H be its decider.

Build a TM S that decides A_{TM}:

S = on input $\langle N, \omega\rangle$

1. Construct TM M_1 on input $\langle x\rangle$:
    1. accept.

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$\boldsymbol{L(M_2) = ?}$

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$\boldsymbol{L(M_2) = \Sigma^*}$

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1. Construct TM M_1 on input $\langle x\rangle$:
    1. accept.
2. Construct TM M_2 on input $\langle y\rangle$:
    1. ?

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$\boldsymbol{N}$ **accepts** $\boldsymbol{\omega}$

$$\Updownarrow$$

$$\boldsymbol{L(M_2) = \Sigma^*}$$

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3. Run H on $\langle M_1, M_2\rangle$.
4. If H accepts, ???

$$\boxed{\begin{array}{c} \boldsymbol{N} \textbf{ accepts } \boldsymbol{\omega} \\ \Updownarrow \\ \boldsymbol{L(M_2) = \Sigma^*} \end{array}}$$

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$\Updownarrow$

$\boldsymbol{L(M_2) = \Sigma^*}$

If N accepts ω, then M_1 and M_2 have the same language (Σ^*) and S accepts. If N does not accept ω, then they have different languages and S rejects. Thus, S decides A_{TM}, which is a contradiction, so EQ_{TM} is undecidable.

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$$\boldsymbol{N \text{ accepts } \omega} \\ \Updownarrow \\ \boldsymbol{L(M_2) = \Sigma^*}$$

Lots of ways to write the recipe!

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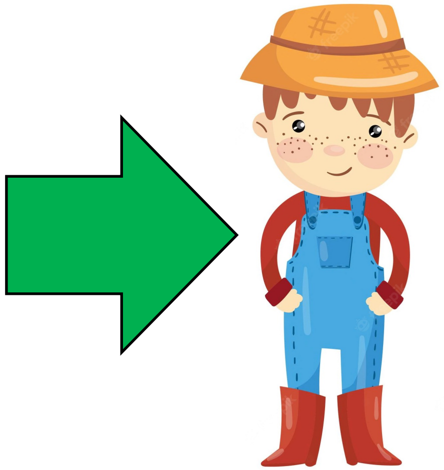

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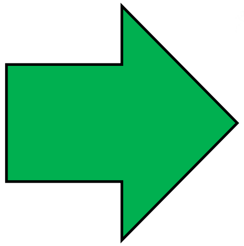
<u>Recipe</u>:
1. If government has aliens, add an egg.



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Recipe:

1. If government has aliens, add an egg.

Won’t Work Recipe:

1. If government has aliens, add flour.
2. Add eggs.

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$\boldsymbol{N}$ **accepts** $\boldsymbol{\omega}$

$\Updownarrow$

$\boldsymbol{L(M_1) = L(M_2)}$

Lots of ways to write the recipe!

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2. Construct TM M_2 on input $\langle y\rangle$:
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3. Run H on $\langle M_1, M_2\rangle$.
4. If H accepts, <u>**reject**</u>. If H rejects, <u>**accept**</u>.

$$\boldsymbol{N} \textbf{ accepts } \boldsymbol{\omega} \Updownarrow \boldsymbol{L(M_1) = L(M_2)}$$

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    2. If N rejects, <u>reject.</u>
    3. <u>accept.</u>
3. Run H on $\langle M_1, M_2\rangle$.
4. If H accepts, <u>accept</u>. If H rejects, <u>reject</u>.

$$\boldsymbol{N \text{ accepts } \omega} \\ \Updownarrow \\ \boldsymbol{L(M_1) = L(M_2)}$$

Lots of ways to write the recipe!