Undecidability
CSCI 338
Claim: $E_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) = \emptyset \}$ is undecidable.

Proof: Suppose $E_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.
   2. Run $H$ on $\langle M_2 \rangle$.
   3. If $H$ accepts, reject. If $H$ rejects, accept.

If $N$ accepts $\omega$, $L(M_2) = \Sigma^*$, $H$ will reject, and $S$ will accept. If $N$ does not accept $\omega$, $L(M_2) = \emptyset$, $H$ will accept, and $S$ will reject. Thus, $S$ decides $A_{TM}$, which is a contradiction, so $E_{TM}$ is undecidable.
\( \mathcal{EQ}_{TM} \)

Claim: \( \mathcal{EQ}_{TM} = \{ \langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \} \) is undecidable.

Proof: Suppose \( \mathcal{EQ}_{TM} \) is decidable and let TM \( H \) be its decider.

Build a TM \( S \) that decides \( A_{TM} \):

\[
S = \text{on input } \langle N, \omega \rangle
\]

1. Construct TM \( M_1 \) on input \( \langle x \rangle \):
   1. accept.
2. Construct TM \( M_2 \) on input \( \langle y \rangle \):
   1. Run \( N \) on \( \omega \) and accept if \( N \) does.
3. Run \( H \) on \( \langle M_1, M_2 \rangle \).
4. If \( H \) accepts, accept. If \( H \) rejects, reject.

If \( N \) accepts \( \omega \), then \( M_1 \) and \( M_2 \) have the same language (\( \Sigma^* \)) and \( S \) accepts. If \( N \) does not accept \( \omega \), then they have different languages and \( S \) rejects. Thus, \( S \) decides \( A_{TM} \), which is a contradiction, so \( \mathcal{EQ}_{TM} \) is undecidable.
Claim: $REGULAR_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular} \}$ is undecidable.

Proof:
Claim: $REGULAR_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular} \}$ is undecidable.

Proof: Suppose $REGULAR_{TM}$ is decidable and let TM $H$ be its decider.
**REGULAR}_{TM}

Claim: \( REGULAR_{TM} = \{ \langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular} \} \) is undecidable.

Proof: Suppose \( REGULAR_{TM} \) is decidable and let TM \( H \) be its decider.

   Build a TM \( S \) that decides \( A_{TM} \):
   
   \[ S = \text{on input } \langle N, \omega \rangle \]
   
   1.
\textit{REGULAR}_{TM}

Claim: $\textit{REGULAR}_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular}\}$ is undecidable.

Proof: Suppose $\textit{REGULAR}_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

1. $S = \text{ on input } \langle N, \omega \rangle$

Plan: Build a TM whose language is regular if $N$ accepts $\omega$ and not regular if $N$ does not accept $\omega$. 
Claim: $\text{REGULAR}_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular}\}$ is undecidable.

Proof: Suppose $\text{REGULAR}_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:

$\text{L}(M_2) \text{ is regular }$ $\iff$ $N \text{ accepts } \omega$

Plan: Build a TM whose language is regular if $N$ accepts $\omega$ and not regular if $N$ does not accept $\omega$. 
Claim: $REGULAR_{TM} = \{ \langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular} \}$ is undecidable.

Proof: Suppose $REGULAR_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S =$ on input $\langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. If $x \in \{ \text{ ??? } \}$, accept.
   2. If $x \notin \{ \text{ ??? } \}$, run $N$ on $\omega$ and accept if $N$ does.

$L(M_2)$ is regular
$\iff$
$N$ accepts $\omega$

Plan: Build a TM whose language is regular if $N$ accepts $\omega$ and not regular if $N$ does not accept $\omega.$
**REGULAR}_{ TM**

Claim: \( \text{REGULAR}_{ TM} = \{ \langle M \rangle: M \text{ is a TM and } L(M) \text{ is regular} \} \) is undecidable.

Proof: Suppose \( \text{REGULAR}_{ TM} \) is decidable and let TM \( H \) be its decider.

Build a TM \( S \) that decides \( A_{ TM} : \)

\[ S = \text{on input } \langle N, \omega \rangle \]

1. Construct TM \( M_2 \) on input \( \langle x \rangle : \)
   1. If \( x \in \{0^n1^n: n \geq 0\} \), accept.
   2. If \( x \notin \{0^n1^n: n \geq 0\} \), run \( N \) on \( \omega \) and accept if \( N \) does.

\( L(M_2) \) is regular \( \iff \)
\( N \) accepts \( \omega \)

Plan: Build a TM whose language is regular if \( N \)
accepts \( \omega \) and not regular if \( N \) does not accept \( \omega \).
Claim: $REGULAR_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular}\}$ is undecidable.

Proof: Suppose $REGULAR_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

- $S = \text{on input } \langle N, \omega \rangle$
  1. Construct TM $M_2$ on input $\langle x \rangle$:
     1. If $x \in \{0^n1^n : n \geq 0\}$, accept.
     2. If $x \notin \{0^n1^n : n \geq 0\}$, run $N$ on $\omega$ and accept if $N$ does.

Plan: Build a TM whose language is regular if $N$ accepts $\omega$ and not regular if $N$ does not accept $\omega$. 

$L(M_2) = ?$
Claim: $\text{REGULAR}_{TM} = \{ \langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular} \}$ is undecidable.

Proof: Suppose $\text{REGULAR}_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. If $x \in \{0^n1^n : n \geq 0\}$, accept.
   2. If $x \not\in \{0^n1^n : n \geq 0\}$, run $N$ on $\omega$ and accept if $N$ does.

Plan: Build a TM whose language is regular if $N$ accepts $\omega$ and not regular if $N$ does not accept $\omega$. 

$L(M_2) = 0^n1^n$ or $\Sigma^*$
Claim: $REGULAR_{TM} = \{\langle M \rangle: M$ is a TM and $L(M)$ is regular$\}$ is undecidable.

Proof: Suppose $REGULAR_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   
   1. If $x \in \{0^n1^n: n \geq 0\}$, accept.
   2. If $x \notin \{0^n1^n: n \geq 0\}$, run $N$ on $\omega$ and accept if $N$ does.

2. ?
Claim: $\text{REGULAR}_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular}\}$ is undecidable.

Proof: Suppose $\text{REGULAR}_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   
   1. If $x \in \{0^n1^n : n \geq 0\}$, accept.
   2. If $x \notin \{0^n1^n : n \geq 0\}$, run $N$ on $\omega$ and accept if $N$ does.

2. Run $H$ on $\langle M_2 \rangle$. 


**REGULAR\textsubscript{TM}**

Claim: \( \text{REGULAR}_{TM} = \{ \langle M \rangle: M \text{ is a TM and } L(M) \text{ is regular} \} \) is undecidable.

Proof: Suppose \( \text{REGULAR}_{TM} \) is decidable and let TM \( H \) be its decider.

Build a TM \( S \) that decides \( A_{TM} : \)

\[ S = \text{on input } \langle N, \omega \rangle \]

1. Construct TM \( M_2 \) on input \( \langle x \rangle : \)
   1. If \( x \in \{ 0^n1^n : n \geq 0 \} \), accept.
   2. If \( x \notin \{ 0^n1^n : n \geq 0 \} \), run \( N \) on \( \omega \) and accept if \( N \) does.

2. Run \( H \) on \( \langle M_2 \rangle \).
3. If \( H \) accepts, accept. If \( H \) rejects, reject.
Claim: $REGULAR_{TM} = \{\langle M \rangle: M \text{ is a TM and } L(M) \text{ is regular}\}$ is undecidable.

Proof: Suppose $REGULAR_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. If $x \in \{0^n1^n: n \geq 0\}$, accept.
   2. If $x \notin \{0^n1^n: n \geq 0\}$, run $N$ on $\omega$ and accept if $N$ does.

2. Run $H$ on $\langle M_2 \rangle$.

3. If $H$ accepts, accept. If $H$ rejects, reject.

If $N$ accepts $\omega$, ... $S$ accepts.
**REGULAR\_TM**

Claim: $REGULAR\_TM = \{\langle M \rangle: M \text{ is a TM and } L(M) \text{ is regular} \}$ is undecidable.

Proof: Suppose $REGULAR\_TM$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A\_TM$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. If $x \in \{0^n1^n: n \geq 0\}$, accept.
   2. If $x \notin \{0^n1^n: n \geq 0\}$, run $N$ on $\omega$ and accept if $N$ does.
2. Run $H$ on $\langle M_2 \rangle$.
3. If $H$ accepts, accept. If $H$ rejects, reject.

If $N$ accepts $\omega$, $L(M_2) = \Sigma^*$ (regular) and $S$ accepts.
\textit{REGULAR}_{TM}

Claim: \(\text{REGULAR}_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular}\} \) is undecidable.

Proof: Suppose \(\text{REGULAR}_{TM}\) is decidable and let TM \(H\) be its decider.

Build a TM \(S\) that decides \(A_{TM}\):

\(S = \) on input \(\langle N, \omega \rangle\)
1. Construct TM \(M_2\) on input \(\langle x \rangle\):
   1. If \(x \in \{0^n1^n : n \geq 0\}\), accept.
   2. If \(x \notin \{0^n1^n : n \geq 0\}\), run \(N\) on \(\omega\) and accept if \(N\) does.
2. Run \(H\) on \(\langle M_2 \rangle\).
3. If \(H\) accepts, accept. If \(H\) rejects, reject.

If \(N\) accepts \(\omega\), \(L(M_2) = \Sigma^*\) (regular) and \(S\) accepts. If \(N\) does not accept \(\omega\), ...
\(S\) rejects.
Claim: $\textit{REGULAR}_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular} \}$ is undecidable.

Proof: Suppose $\textit{REGULAR}_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S$ = on input $\langle N, \omega \rangle$

1. Construct TM $M_2$ on input $\langle \omega \rangle$:
   1. If $\omega \in \{0^n1^n : n \geq 0\}$, accept.
   2. If $\omega \notin \{0^n1^n : n \geq 0\}$, run $N$ on $\omega$ and accept if $N$ does.

2. Run $H$ on $\langle M_2 \rangle$.

3. If $H$ accepts, accept. If $H$ rejects, reject.

If $N$ accepts $\omega$, $L(M_2) = \Sigma^*$ (regular) and $S$ accepts. If $N$ does not accept $\omega$, $L(M_2) = \{0^n1^n : n \geq 0\}$ (not regular) and $S$ rejects.
\textbf{REGULAR}_{TM}

Claim: \(\text{REGULAR}_{TM} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular}\}\) is undecidable.

Proof: Suppose \(\text{REGULAR}_{TM}\) is decidable and let TM \(H\) be its decider.

Build a TM \(S\) that decides \(A_{TM}\):

\(S = \text{on input } \langle N, \omega \rangle\)

1. Construct TM \(M_2\) on input \(\langle x \rangle\):
   
   1. If \(x \in \{0^n1^n : n \geq 0\}\), \underline{accept}.
   2. If \(x \notin \{0^n1^n : n \geq 0\}\), run \(N\) on \(\omega\) and \underline{accept} if \(N\) does.

2. Run \(H\) on \(\langle M_2 \rangle\).

3. If \(H\) accepts, \underline{accept}. If \(H\) rejects, \underline{reject}.

If \(N\) accepts \(\omega\), \(L(M_2) = \Sigma^*\) (regular) and \(S\) accepts. If \(N\) does not accept \(\omega\), \(L(M_2) = \{0^n1^n : n \geq 0\}\) (not regular) and \(S\) rejects. Therefore, \(S\) is a decider for \(A_{TM}\), which is a contradiction, so \(\text{REGULAR}_{TM}\) is undecidable.
Claim: $EQ_{TM} = \{(A, B): A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $A_{TM}$:

$S = \text{on input } \langle N, \omega \rangle$

1. Construct TM $M_1$ on input $\langle x \rangle$:
   1. accept.

2. Construct TM $M_2$ on input $\langle y \rangle$:
   1. Run $N$ on $\omega$ and accept if $N$ does.

3. Run $H$ on $\langle M_1, M_2 \rangle$.

4. If $H$ accepts, accept. If $H$ rejects, reject.

If $N$ accepts $\omega$, then $M_1$ and $M_2$ have the same language ($\Sigma^*$) and $S$ accepts. If $N$ does not accept $\omega$, then they have different languages and $S$ rejects. Thus, $S$ decides $A_{TM}$, which is a contradiction, so $EQ_{TM}$ is undecidable.
$EQ_{TM}$

Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

   Build a TM $S$ that decides $E_{TM}$:
   
   $S = \text{on input } \langle P \rangle$
   
   1. 
Claim: $EQ_{TM} = \{(A, B): A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $E_{TM}$:

$S =$ on input $\langle P \rangle$

1. We have a way ($H$) to test if two TMs have the same language. How could we use that to test if a TM’s language is empty? Plan: ?
Claim: $E_{Q_{TM}} = \{(A, B) : A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $E_{Q_{TM}}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $E_{TM}$:

$S = \text{ on input } \langle P \rangle$

1. We have a way ($H$) to test if two TMs have the same language. How could we use that to test if a TM’s language is empty?

Plan: Make a TM with an empty language and use $H$ to compare it to input to $E_{TM}$. 
Claim: $EQ_{TM} = \{ \langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $E_{TM}$:

$S = \text{on input } \langle P \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. reject.
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $E_{TM}$:

$S = \text{on input } \langle P \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   
   1. reject.

$L(M_2) = ?$
Claim: $EQ_{TM} = \{\langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $E_{TM}$:

$S = $ on input $\langle P \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. reject.

$L(M_2) = \emptyset$
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $E_{TM}$:

$S = \text{on input } \langle P \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. reject.

2. ?
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B$ are TMs and $L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $E_{TM}$:

$S = $ on input $\langle P \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   
   1. reject.

2. Run $H$ on $\langle P, M_2 \rangle$. 
Claim: \( EQ_{TM} = \{ \langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \} \) is undecidable.

Proof: Suppose \( EQ_{TM} \) is decidable and let TM \( H \) be its decider.

Build a TM \( S \) that decides \( E_{TM} \):

\[
S = \text{on input } \langle P \rangle \\
\quad 1. \text{Construct TM } M_2 \text{ on input } \langle x \rangle : \\
\quad \quad 1. \text{ reject.} \\
\quad 2. \text{Run } H \text{ on } \langle P, M_2 \rangle . \\
\quad 3. \text{If } H \text{ accepts, } \_\text{?}\_. \text{ If } H \text{ rejects, } \_\text{?}\_.
\]
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $E_{TM}$:

$S = \text{on input } \langle P \rangle$

1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. reject.
   2. Run $H$ on $\langle P, M_2 \rangle$.
   3. If $H$ accepts, accept. If $H$ rejects, reject.
Claim: \( EQ_{TM} = \{ \langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \} \) is undecidable.

Proof: Suppose \( EQ_{TM} \) is decidable and let TM \( H \) be its decider.

Build a TM \( S \) that decides \( E_{TM} \):

\[ S = \text{on input } \langle P \rangle \]

1. Construct TM \( M_2 \) on input \( \langle x \rangle \):
   1. reject.
   2. Run \( H \) on \( \langle P, M_2 \rangle \).
   3. If \( H \) accepts, accept. If \( H \) rejects, reject.

If \( L(P) = \emptyset \), ... \( S \) will accept. If \( L(P) \neq \emptyset \), ... \( S \) will reject.
Claim: \( EQ_{TM} = \{ \langle A, B \rangle : A, B \text{ are TMs and } L(A) = L(B) \} \) is undecidable.

Proof: Suppose \( EQ_{TM} \) is decidable and let TM \( H \) be its decider.

Build a TM \( S \) that decides \( E_{TM} \):

\[
S = \text{on input } \langle P \rangle \\
1. \text{Construct TM } M_2 \text{ on input } \langle x \rangle : \\
   \quad 1. \text{ reject.} \\
2. \text{Run } H \text{ on } \langle P, M_2 \rangle. \\
3. \text{If } H \text{ accepts, accept. If } H \text{ rejects, reject.}
\]

If \( L(P) = \emptyset \), \( M_2 \) and \( P \) will have the same language (since \( L(M_2) = \emptyset \)) and \( S \) will accept. If \( L(P) \neq \emptyset \), ... \( S \) will reject.
Claim: \( EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\} \) is undecidable.

Proof: Suppose \( EQ_{TM} \) is decidable and let TM \( H \) be its decider.

Build a TM \( S \) that decides \( E_{TM} \):

\[
S = \text{on input } \langle P \rangle \\
1. \text{Construct TM } M_2 \text{ on input } \langle x \rangle : \\
   1. \text{ reject.} \\
2. \text{ Run } H \text{ on } \langle P, M_2 \rangle. \\
3. \text{ If } H \text{ accepts, accept. If } H \text{ rejects, reject.}
\]

If \( L(P) = \emptyset \), \( M_2 \) and \( P \) will have the same language (since \( L(M_2) = \emptyset \)) and \( S \) will accept. If \( L(P) \neq \emptyset \), \( M_2 \) and \( P \) will not have the same language and \( S \) will reject.
Claim: $EQ_{TM} = \{\langle A, B \rangle: A, B \text{ are TMs and } L(A) = L(B)\}$ is undecidable.

Proof: Suppose $EQ_{TM}$ is decidable and let TM $H$ be its decider.

Build a TM $S$ that decides $E_{TM}$:

$S = \text{ on input } \langle P \rangle$
1. Construct TM $M_2$ on input $\langle x \rangle$:
   1. reject.
2. Run $H$ on $\langle P, M_2 \rangle$.
3. If $H$ accepts, accept. If $H$ rejects, reject.

If $L(P) = \emptyset$, $M_2$ and $P$ will have the same language (since $L(M_2) = \emptyset$) and $S$ will accept. If $L(P) \neq \emptyset$, $M_2$ and $P$ will not have the same language and $S$ will reject. Therefore, $S$ is a decider for $E_{TM}$, which is a contradiction, so $EQ_{TM}$ is undecidable.