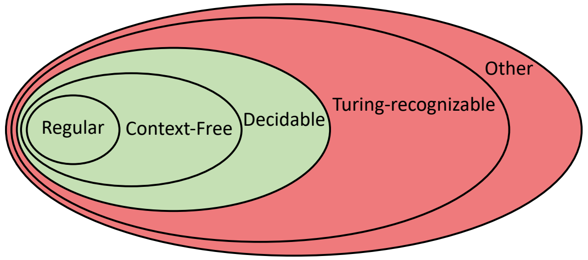
# P and NP
## CSCI 338

# Announcements

- Test 2. (AVG = 72%, MED = 85%)
- Project 2. (AVG = 80%, MED = 100%)

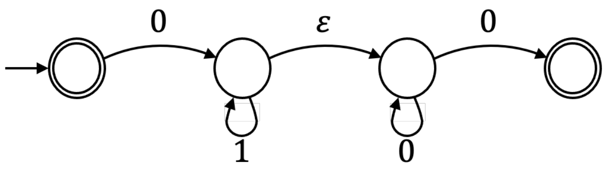
Computability: What’s solvable by computers.



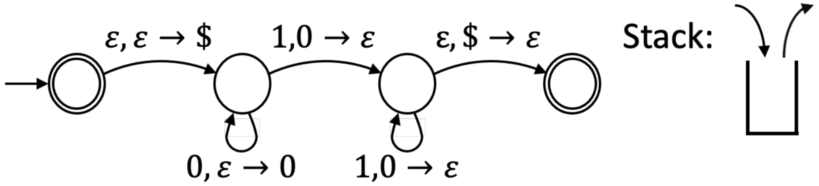

**January** **May**

Computational Models

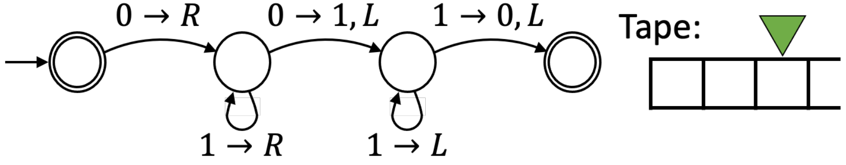
NFA




PDA




Turing Machine




Complexity: What’s efficiently solvable by computers.

Goal: Understand and identify fundamental limitations of computers.

# Computational Complexity

Determining the amount of resources required to accomplish some task (solve a problem).

- Time (most common)
- Space (close behind)
- Power (sensor networks, spacecraft, military)
- Network (Netflix, IoT)

# Definitions

Definition: For TM $M$, the **running time of $\boldsymbol{M}$** is the function $f: \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum number of steps $M$ used on any input of length $n$.

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Definition: For TM $M$, the **running time of $\boldsymbol{M}$** is the function $f\colon \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum number of steps $M$ used on any input of length $n$.

Definition: Let $f$ and $g$ be functions $f, g\colon \mathbb{N} \to \mathbb{R}^+$, $\boldsymbol{f(n) \in O\big(g(n)\big)}$ if $\exists$ positive integers $c$ and $n_0$ such that $\forall\, n \geq n_0$, $f(n) \leq c\, g(n)$.

Asymptotic upper bound on running time.

# P

Definition: Let $t: \mathbb{N} \rightarrow \mathbb{R}^{+}$ be a function. The time complexity class, **TIME($\boldsymbol{t(n)}$)**, is the collection of all languages that are decidable by an $O(t(n))$ time TM.

Definition: $\boldsymbol{P}$ is the set of languages that are decidable in polynomial time on a deterministic single-tape TM.

$$P = \bigcup_{k} \text{TIME}\left(n^{k}\right)$$

Properties of $P$:

- Inclusion in $P$ holds for all computational models polynomially equivalent to deterministic, single-tape TMs.
- Roughly corresponds to problems solvable by a computer.

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Properties of $P$:

- Inc… po… e TMs.
- Roughly corresponds to problems solvable by a computer.

**How do you show something is in $\boldsymbol{P}$?**

**Build a polynomial time decider for it.**

# Why Polynomial?

Given a problem of size $n$...

Algorithm A solves it in $n^2$ seconds

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A solves in $1$ min $\Rightarrow$ A solves in $60$ sec $\Rightarrow n \approx 7.7$

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B solves in 5 min $\Rightarrow$ B solves in 300 sec $\Rightarrow n \approx 8.2$

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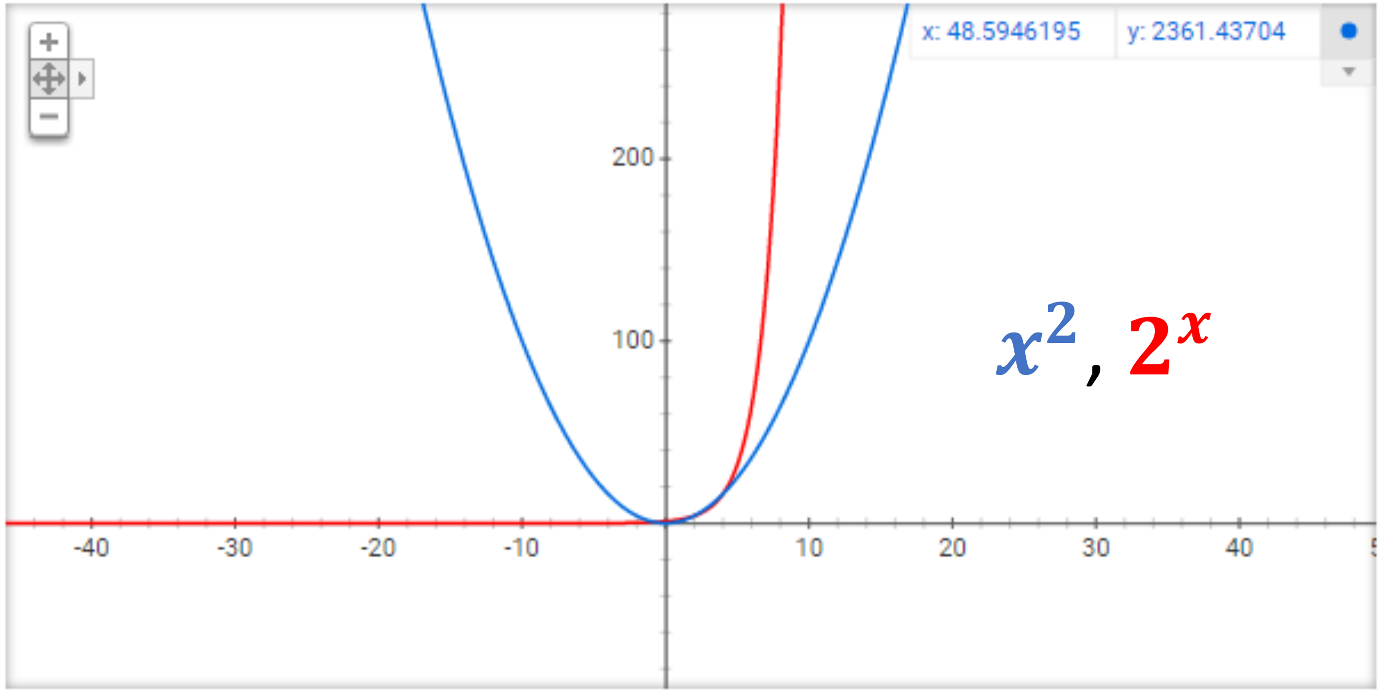
A solves in $1$ hr $\Rightarrow$ A solves in $3600$ sec $\Rightarrow n = 60$

B solves in $10\text{B}$ yr $\Rightarrow$ B solves in lots of sec $\Rightarrow n \approx 58$

315360000000000000

# Why Polynomial?




$x^2$, $2^x$

Polynomial time algorithms are usually useful,
exponential time algorithms are rarely useful

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1. Mark $s$.
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At most, how many edges are in an undirected graph with $n$ vertices?
What graph has the most number of edges?

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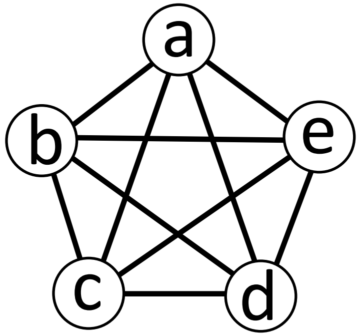


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How many edges does a complete graph with $n$ vertices have?

How many edges leave each vertex?

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How much does that all add up to?

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How much does that all add up to? $n(n - 1)$

Did we double count any edges?

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So how many edges are there?

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What if it is directed? $n(n-1) \in O(n^2)$ (no double counts)

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N is a decider and runs in $O(n^3)$ time, therefore $PATH \in P$.