NP
CSCI 338
\( \mathcal{NP} \)

\( P \) is the set of languages that are solvable (decidable) in polynomial time.
To show something is in \( P \), build a polynomial time decider for it.
$NP$

$P$ is the set of languages that are solvable (decidable) in polynomial time.

To show something is in $P$, build a polynomial time decider for it.

$NP \left\{ \text{Set of languages that have polynomial time verifiers.} \right\}$
Vertex Cover (VC)

Vertex Cover = \{(G, k): G = (V, E) is a graph and k is an integer \leq |V| such that there exists some V' \subseteq V with |V'| \leq k, such that each edge in E contains an end point in V'\}
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Is there a VC \( \leq k \) for \( k = 8 \)?
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[Graph of a vertex cover problem]
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Is there a VC \( \leq k \) for \( k = 5 \)?
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Is there a \( VC \leq k \) for \( k = 4 \)?
Vertex Cover (VC)

Vertex Cover = \{ (G, k) : G = (V, E) is a graph and k is an integer \leq |V| such that there exists some V' \subseteq V with |V'| \leq k, such that each edge in E contains an end point in V' \}\n
Vertex Cover: Given graph G = (V, E) and integer k \leq |V|, is there V' \subseteq V, with |V'| \leq k, such that each edge in E contains an end point in V'?

Is there a VC \leq k for k = 4?
Vertex Cover (VC)

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Vertex Cover: Given graph G = (V, E) and integer k ≤ |V|, is there V′ ⊆ V, with |V′| ≤ k, such that each edge in E contains an end point in V′?

Is there a VC ≤ k for k = 4?
Vertex Cover (VC)

Vertex Cover = \{\langle G, k \rangle: G = (V, E) is a graph and k is an integer \leq |V| such that there exists some \( V' \subseteq V \) with |\( V' \)| \leq k, such that each edge in \( E \) contains an end point in \( V' \) \}

Vertex Cover: Given graph \( G = (V, E) \) and integer \( k \leq |V| \), is there \( V' \subseteq V \), with |\( V' \)| \leq k, such that each edge in \( E \) contains an end point in \( V' \)?

Is there a VC \( \leq k \) for \( k = 4 \)?

Decision problem:
“Yes/No” – Is there a VC \( \leq k \)?

Optimization problem:
“Best” – What is the smallest VC?
Vertex Cover (VC)

Claim: VC ∈ NP

Proof:

Decider: Is ⟨G, k⟩ ∈ VC?
Vertex Cover (VC)

Claim: VC ∈ NP

Proof:

Decider: Is \( \langle G, k \rangle \in VC \)?

Verifier: Is \( \langle G, k \rangle \in VC \), given a candidate solution?
Vertex Cover (VC)

Claim: VC ∈ NP

Proof:

Decider: Is \( \langle G, k \rangle \in VC \)?

Verifier: Is \( \langle G, k \rangle \in VC \), given a candidate solution?
Vertex Cover (VC)

Claim: VC $\in \mathcal{NP}$

Proof:
Build a polynomial time verifier.

Vertex Cover: Given graph $G = (V, E)$ and integer $k \leq |V|$, is there $V' \subseteq V$, with $|V'| \leq k$, such that each edge in $E$ contains an end point in $V'$?
Vertex Cover (VC)

Claim: VC \( \in \mathcal{NP} \)

Proof:

Build a polynomial time verifier.

\[ M = \text{on input } \langle G, k, V' \rangle, \text{ where } V' \text{ is a subset of } V. \]

1. ???.
Vertex Cover (VC)

Claim: VC ∈ \( NP \)

Proof:

Build a polynomial time verifier.

\[ M = \text{on input } \langle G, k \rangle, V' \rangle, \text{ where } V' \text{ is a subset of } V. \]

1. Test if \(|V'| \leq k\), reject if not.
Vertex Cover (VC)

Claim: VC $\in \mathcal{NP}$

Proof:

Build a polynomial time verifier.

$M = \text{on input } \langle G, k \rangle, V', \text{ where } V' \subseteq V.$

1. Test if $|V'| \leq k$, reject if not.

2. ???
Vertex Cover (VC)

Claim: \( VC \in NP \)

Proof:

Build a polynomial time verifier.

\[ M = \text{on input } \langle G, k \rangle, V' \rangle, \text{ where } V' \text{ is a subset of } V. \]

1. Test if \( |V'| \leq k \), reject if not.

2. For each edge \( e = (a, b) \) in \( E \),

   2.1 ????
Vertex Cover (VC)

Claim: VC ∈ NP

Proof:

Build a polynomial time verifier.

\[ M = \text{on input } \langle \langle G, k \rangle, V' \rangle, \text{ where } V' \text{ is a subset of } V. \]

1. Test if \(|V'| \leq k\), reject if not.

2. For each edge \(e = (a, b)\) in \(E\),
   2.1 Test if \(a \in V'\) or \(b \in V'\), ???.

Vertex Cover: Given graph \(G = (V, E)\) and integer \(k \leq |V|\), is there \(V' \subseteq V\), with \(|V'| \leq k\), such that each edge in \(E\) contains an end point in \(V'\)?
Vertex Cover (VC)

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Build a polynomial time verifier.

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   2.1 Test if \( a \in V' \) or \( b \in V' \), reject if neither.
Vertex Cover (VC)

Claim: VC $\in NP$

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Build a polynomial time verifier.

$M = \text{on input } \langle G, k \rangle, V' \rangle$, where $V'$ is a subset of $V$.

1. Test if $|V'| \leq k$, reject if not.
2. For each edge $e = (a, b)$ in $E$,
   2.1 Test if $a \in V'$ or $b \in V'$, reject if neither.
3. ???.
Vertex Cover (VC)

Claim: VC ∈ NP

Proof:

Build a polynomial time verifier.

\[ M = \text{on input } \langle \langle G, k \rangle, V' \rangle, \text{ where } V' \text{ is a subset of } V. \]

1. Test if \(|V'| \leq k\), reject if not.
2. For each edge \(e = (a, b)\) in \(E\),
   2.1 Test if \(a \in V'\) or \(b \in V'\), reject if neither.
3. accept.
Vertex Cover (VC)

Claim: VC ∈ NP

Proof:

Build a polynomial time verifier.

$|V| = n.$

$M = \text{on input } \langle G, k \rangle, V', \text{ where } V' \text{ is a subset of } V.$

$O(\cdot) \rightarrow 1. \text{ Test if } |V'| \leq k, \text{ reject if not.}$

2. For each edge $e = (a, b) \in E$,

2.1 Test if $a \in V'$ or $b \in V'$, reject if neither.

3. accept.
Vertex Cover (VC)

Claim: VC ∈ NP

Proof:

Build a polynomial time verifier.

$|V| = n.$

$M =$ on input $⟨⟨G, k⟩, V'⟩$, where $V'$ is a subset of $V$.

$O(1)$

1. Test if $|V'| \leq k$, reject if not.

2. For each edge $e = (a, b)$ in $E$,

   2.1 Test if $a \in V'$ or $b \in V'$, reject if neither.

3. accept.

Vertex Cover: Given graph $G = (V, E)$ and integer $k \leq |V|$, is there $V' \subseteq V$, with $|V'| \leq k$, such that each edge in $E$ contains an end point in $V'$?
Vertex Cover (VC)

Claim: VC ∈ NP

Proof:

Build a polynomial time verifier.

|V| = n.

\[ M = \text{on input } \langle (G, k), V' \rangle, \text{ where } V' \text{ is a subset of } V. \]

\[ O(1) \rightarrow 1. \] Test if \(|V'| \leq k\), reject if not.

\[ O(?) \rightarrow 2. \] For each edge \(e = (a, b)\) in \(E\),

\[ 2.1 \] Test if \(a \in V'\) or \(b \in V'\), reject if neither.

3. accept.
**Vertex Cover (VC)**

**Claim:** $\text{VC} \in \mathcal{NP}$

**Proof:**

1. **Test if** $|V'| \leq k$, **reject** if not.
2. For each edge $e = (a, b)$ in $E$, **2.1 Test if** $a \in V'$ or $b \in V'$, **reject** if neither.
3. **accept**.

**Vertex Cover:**
Given graph $G = (V, E)$ and integer $k \leq |V|$, is there $V' \subseteq V$, with $|V'| \leq k$, such that each edge in $E$ contains an end point in $V'$?

At most, how many edges are in an undirected graph with $n$ vertices?

What graph has the most number of edges?

Complete graph (every pair of vertices have an edge).

How many edges does a complete graph with $n$ vertices have?

How many edges leave each vertex? $n - 1$

How much does that all add up to? $n(n - 1)$

Did we double count any edges? Yes

So how many edges are there? $\frac{n(n-1)}{2} \in O(n^2)$

What if it is directed? $n(n - 1) \in O(n^2)$ (no double counts)
Vertex Cover (VC)

Claim: VC ∈ \( NP \)

Proof:

Build a polynomial time verifier.

\(|V| = n.\)

\( M = \) on input \( \langle G, k \rangle, V' \), where \( V' \) is a subset of \( V \).

\( O(1) \rightarrow 1. \) Test if \( |V'| \leq k \), reject if not.

\( O(n^2) \rightarrow 2. \) For each edge \( e = (a, b) \) in \( E \),

\( \quad 2.1 \) Test if \( a \in V' \) or \( b \in V' \), reject if neither.

3. accept.
Vertex Cover (VC)

Claim: VC \in NP

Proof:

Build a polynomial time verifier.

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Build a polynomial time verifier.

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$O(n) \rightarrow 2.1 \text{ Test if } a \in V' \text{ or } b \in V', \text{ reject if neither.}$

3. accept.
Vertex Cover (VC)

Claim: VC $\in NP$

Proof:

Build a polynomial time verifier.

$|V| = n$. $M = \text{on input } \langle G, k \rangle, V'$, where $V'$ is a subset of $V$.

$O(1) \rightarrow 1. \text{ Test if } |V'| \leq k, \text{ reject if not.}$

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$O(n) \rightarrow 2.1 \text{ Test if } a \in V' \text{ or } b \in V', \text{ reject if neither.}$

$O(?) \rightarrow 3. \text{ accept.}$
Claim: \( \text{VC} \in \mathcal{NP} \)

Proof:

Build a polynomial time verifier.

1. Test if \( |V'| \leq k \), reject if not.

2. For each edge \( e = (a, b) \) in \( E \),

2.1 Test if \( a \in V' \) or \( b \in V' \), reject if neither.

3. accept.

\[ |V| = n. \]

\[ M = \text{on input } \langle \langle G, k \rangle, V' \rangle, \text{ where } V' \text{ is a subset of } V. \]

\[ O(1) \rightarrow 1. \] Test if \( |V'| \leq k \), reject if not.

\[ O(n^2) \rightarrow 2. \] For each edge \( e = (a, b) \) in \( E \),

\[ O(n) \rightarrow 2.1 \] Test if \( a \in V' \) or \( b \in V' \), reject if neither.

\[ O(1) \rightarrow 3. \] accept.

Vertex Cover: Given graph \( G = (V, E) \) and integer \( k \leq |V| \), is there \( V' \subseteq V \), with \( |V'| \leq k \), such that each edge in \( E \) contains an end point in \( V' \)?
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Claim: VC ∈ NP

Proof:

Build a polynomial time verifier.

$|V| = n$. $M = \text{on input } \langle G, k \rangle, V', \text{ where } V' \text{ is a subset of } V$.

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$O(n) \rightarrow 2.1 \text{ Test if } a \in V' \text{ or } b \in V', \text{ reject if neither.}$

$O(1) \rightarrow 3. \text{ accept.}$

For $|V| = n$, $M$ runs in $O(n^3)$ time, therefore $VC \in NP$. 