NP

CSCI 338

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NP — Set of languages that have polynomial time verifiers.

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Vertex Cover = $\{\langle G, k\rangle : G = (V, E)$ is a graph and k is an integer $\leq |V|$ such that there exists some $V' \subseteq V$ with $|V'| \leq k,$ such that each edge in E contains an end point in $V'\}$

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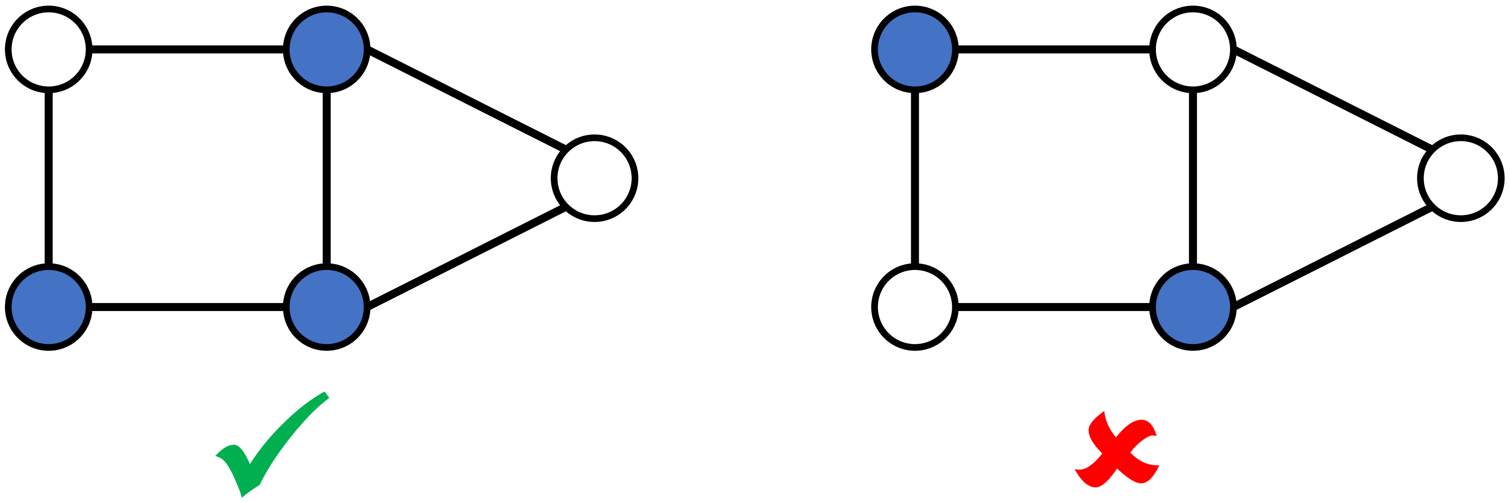

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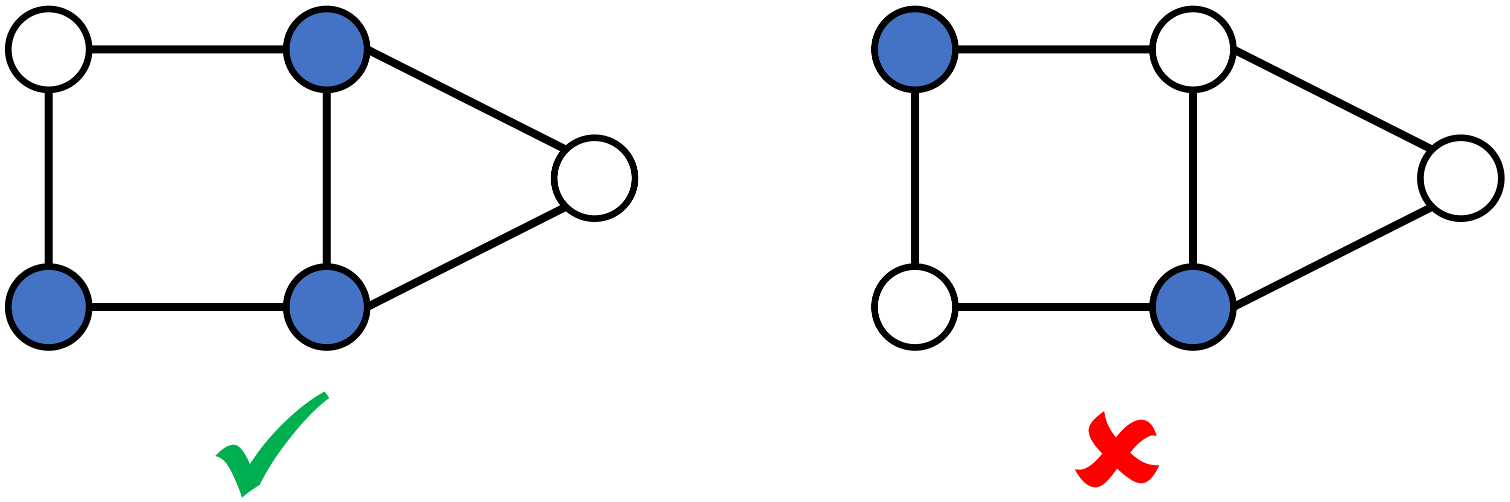

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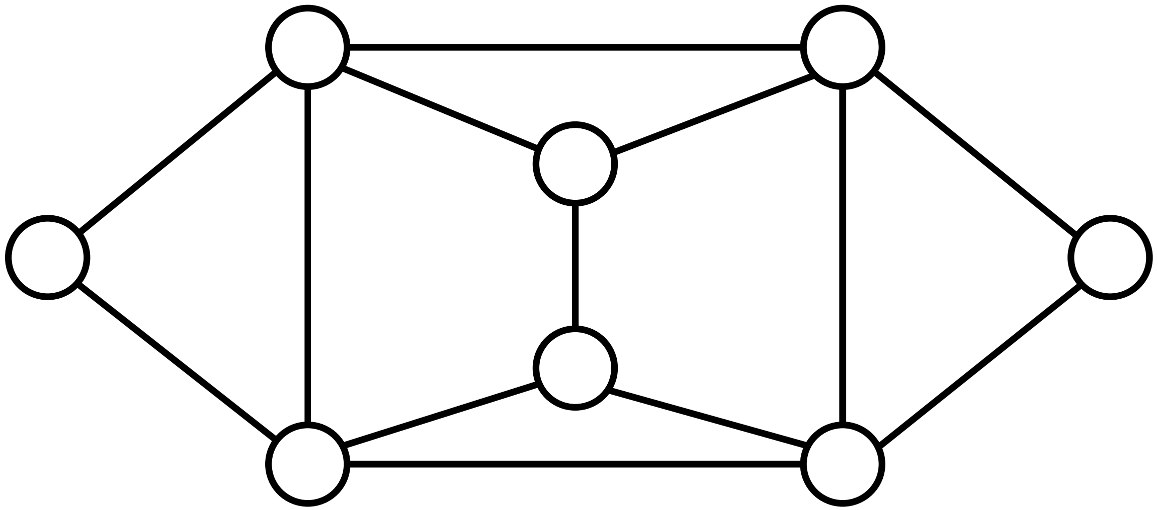

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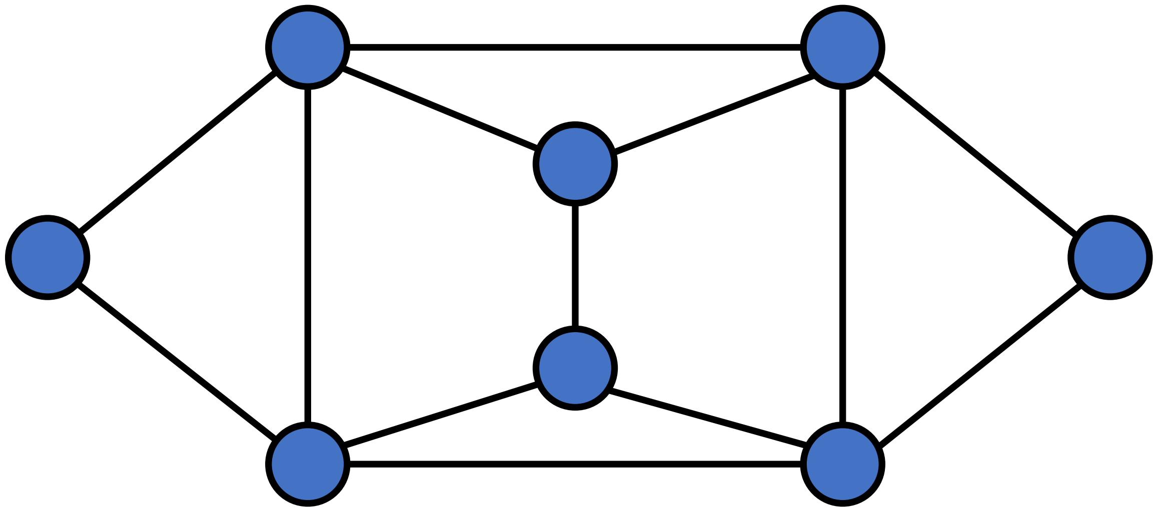

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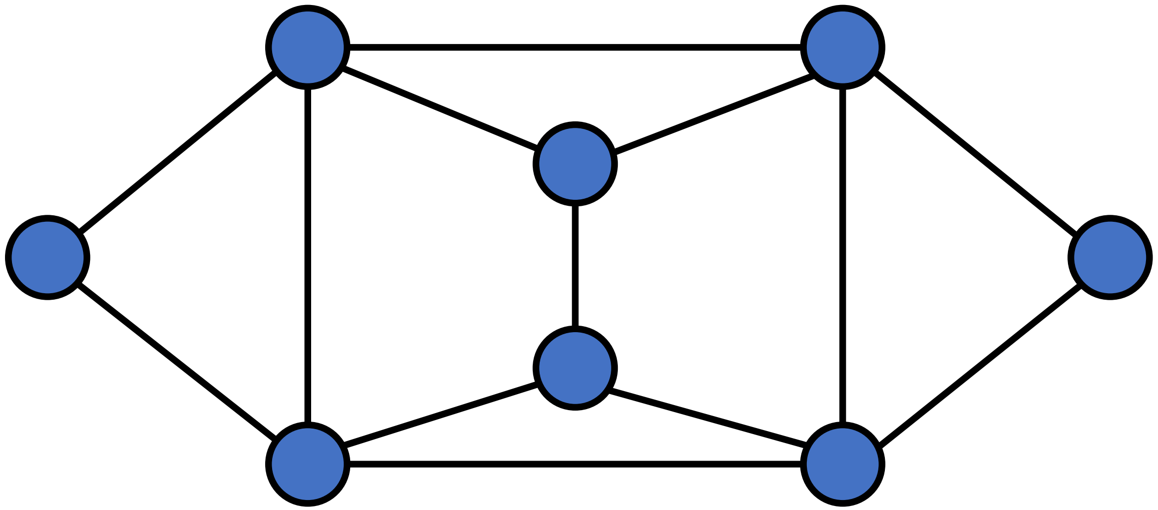

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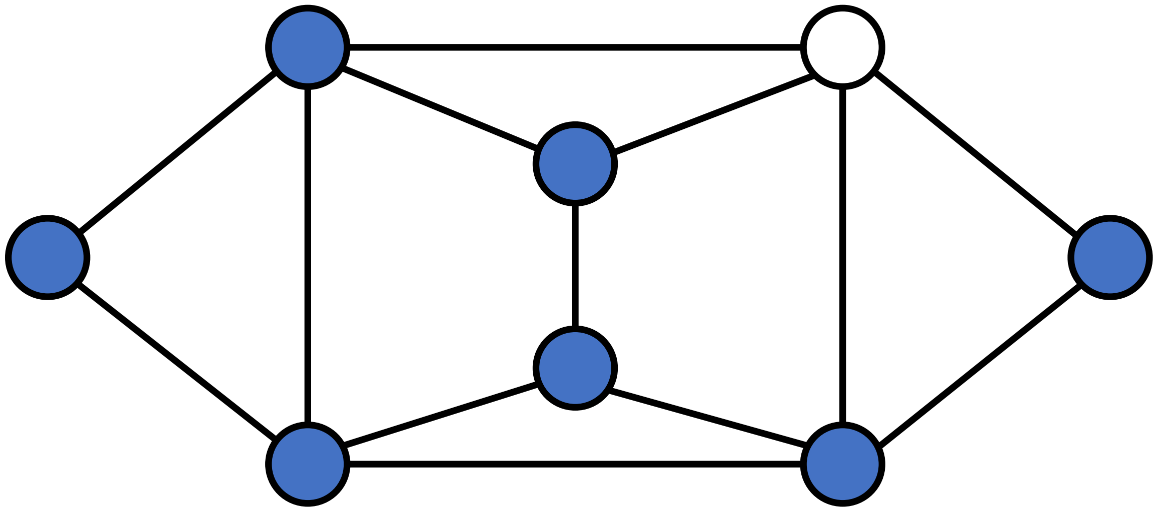

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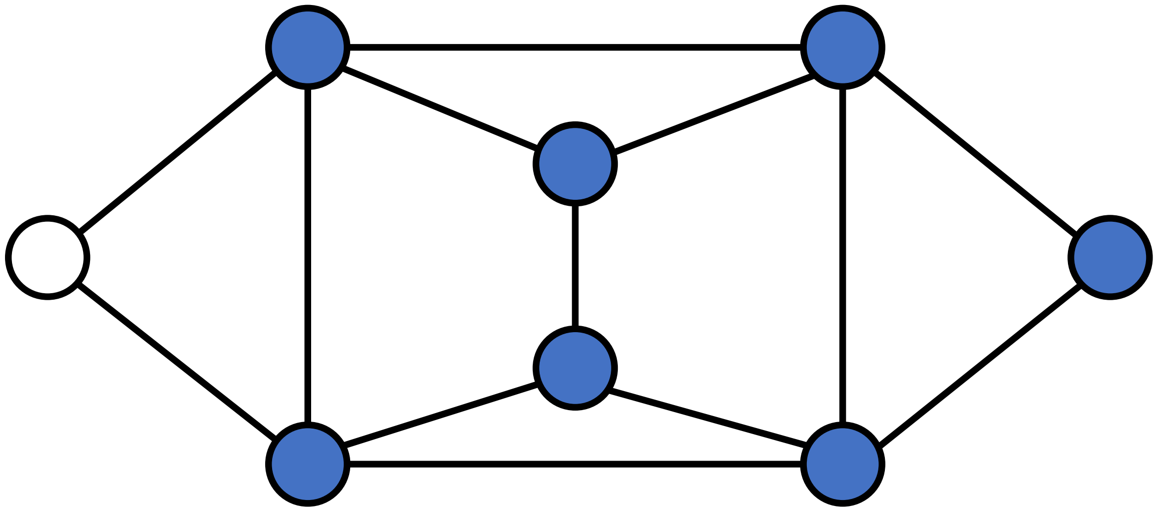

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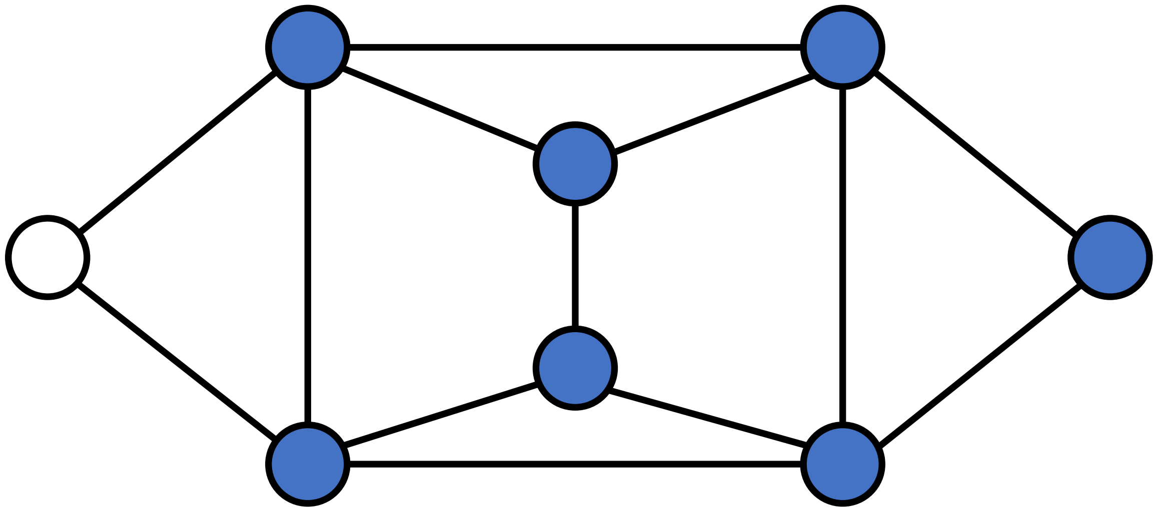

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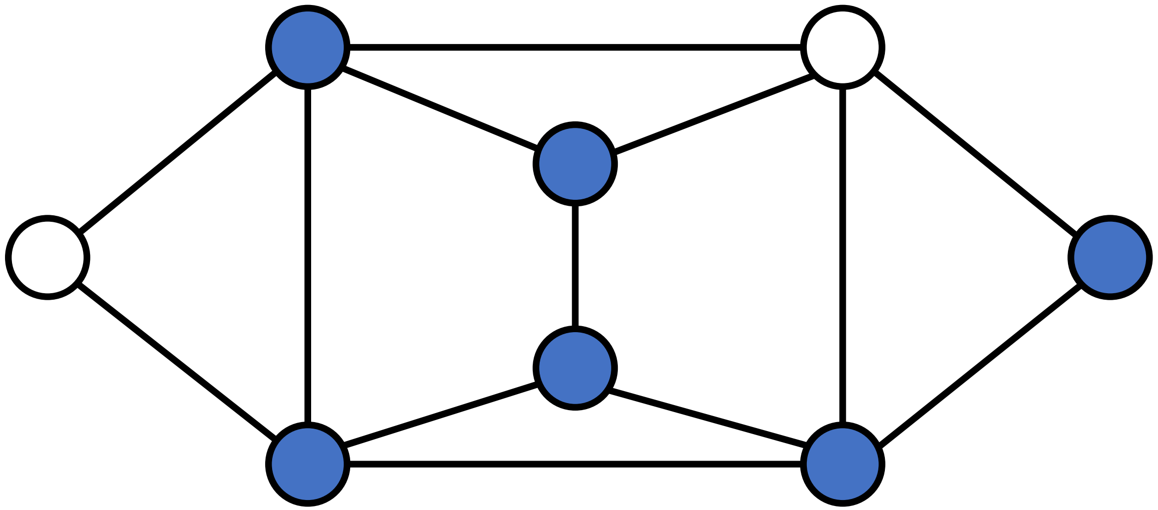

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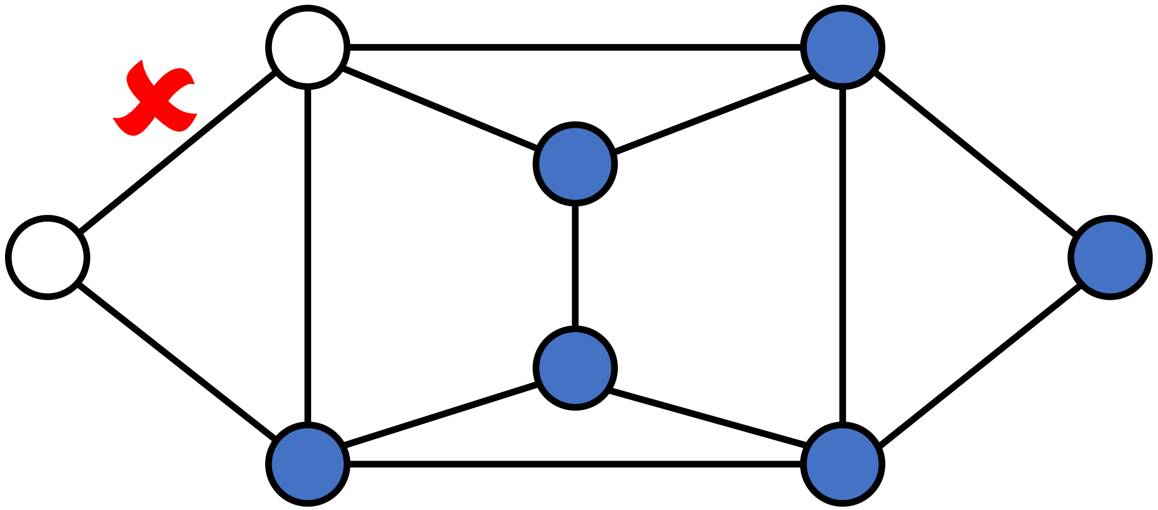

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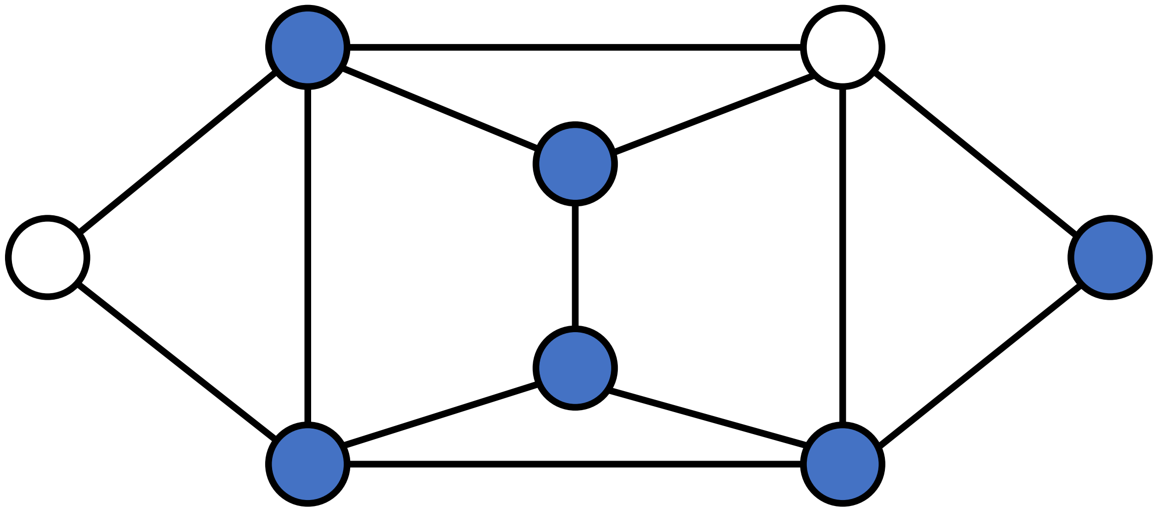

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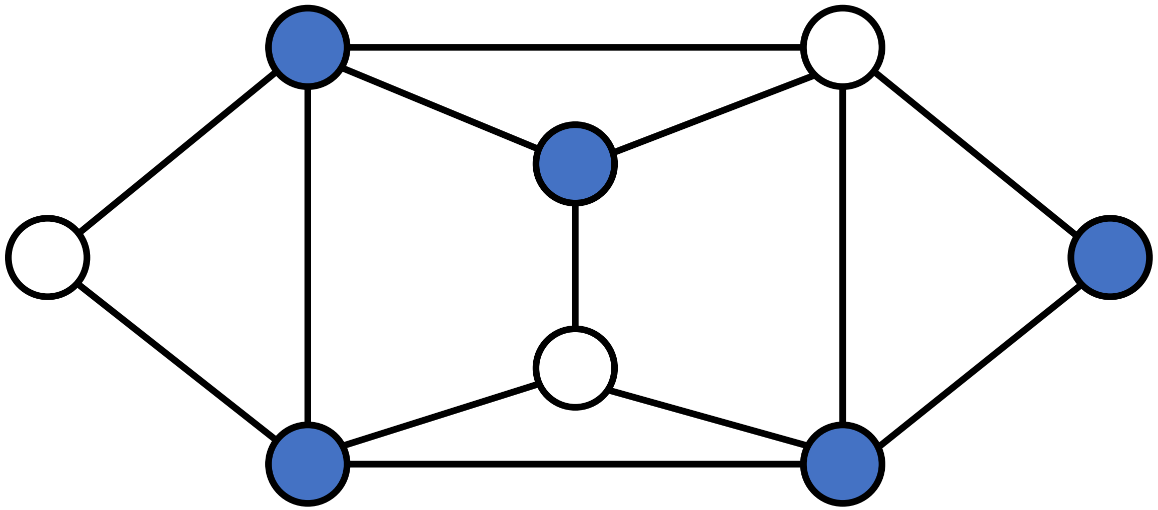

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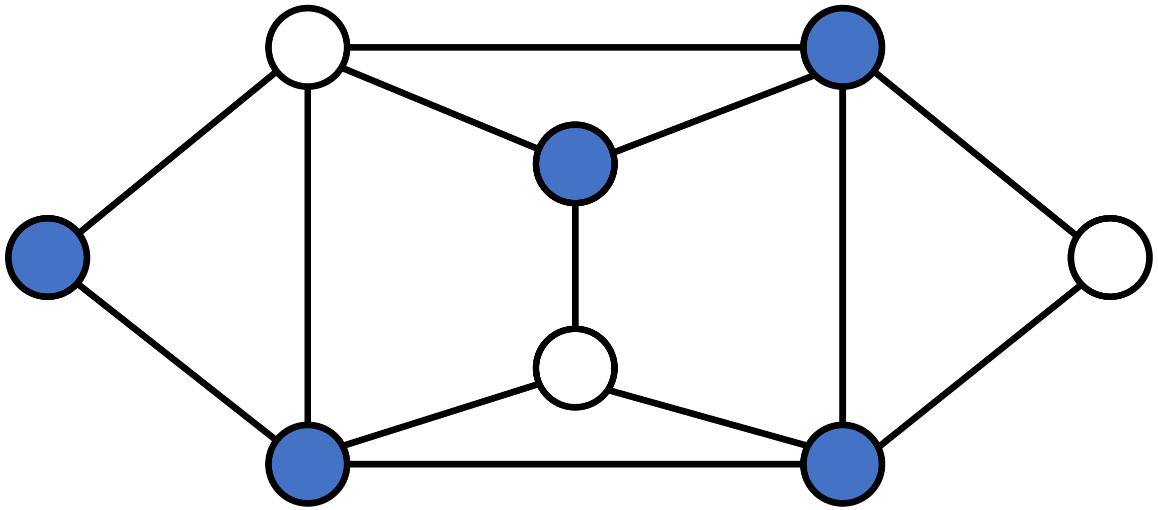

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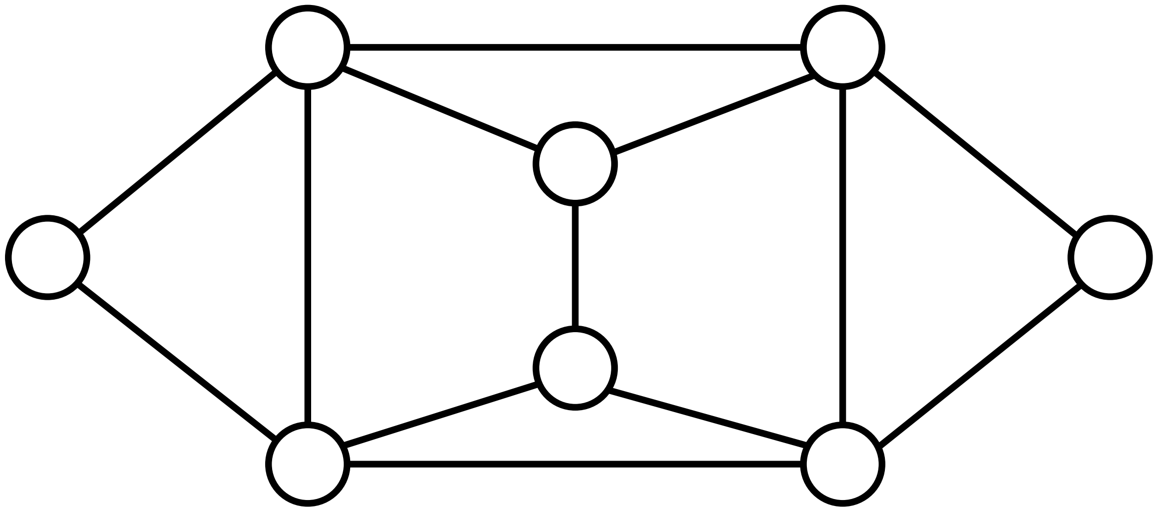

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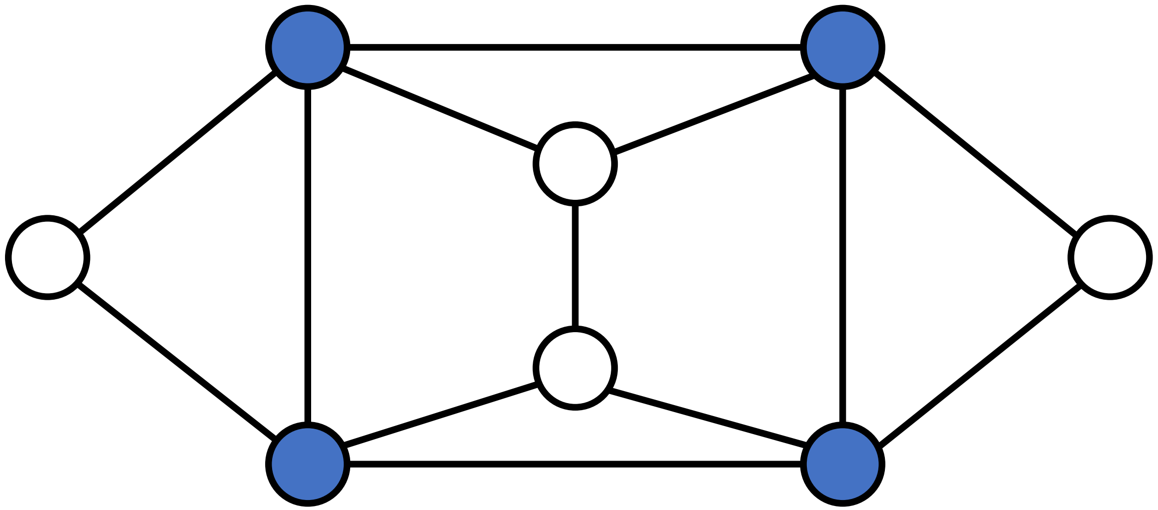

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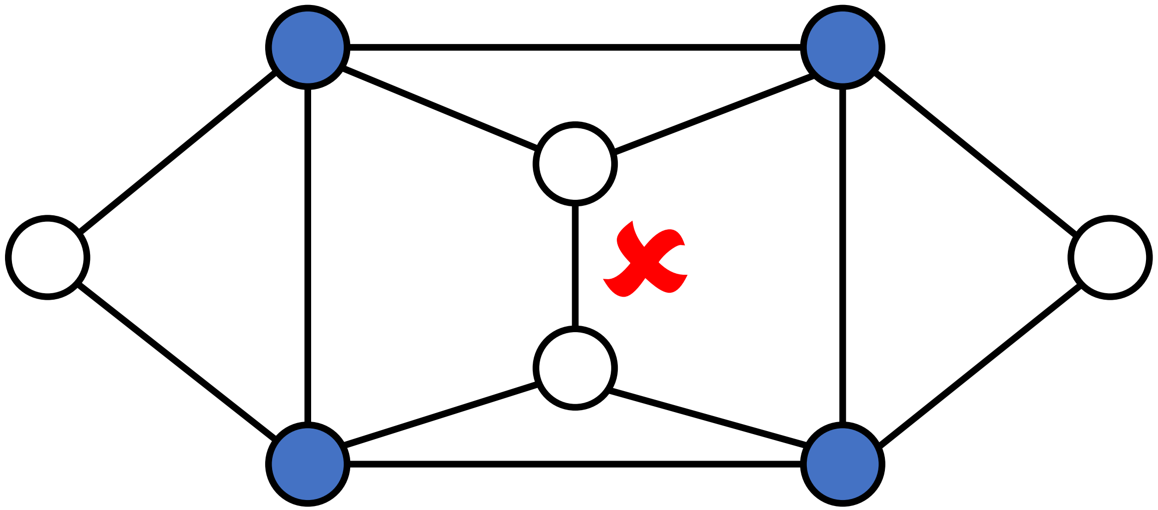

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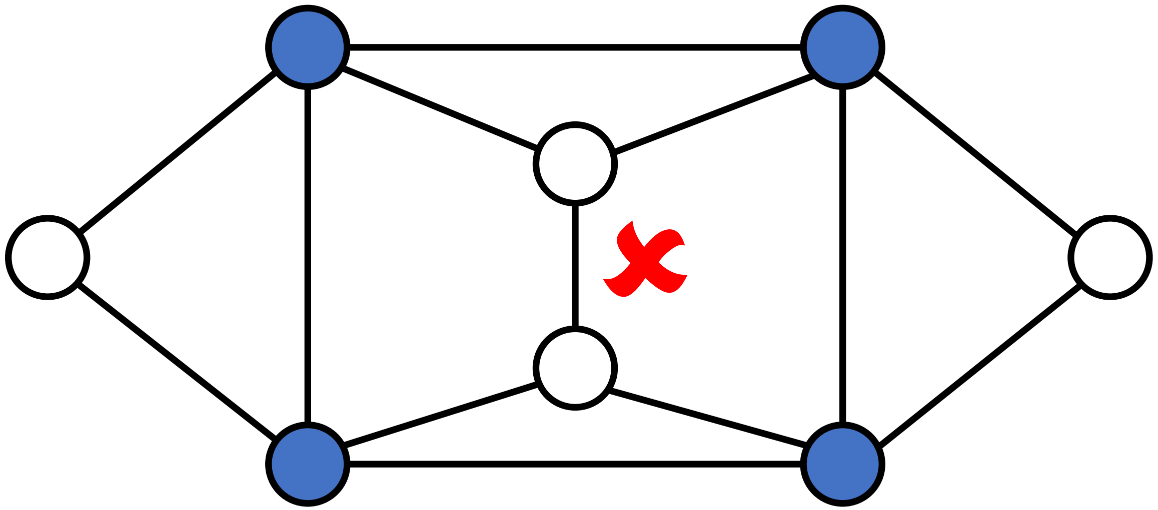

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Decision problem:

"Yes/No" – Is there a VC $\leq k$?

Optimization problem:

"Best" – What is the smallest VC?

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Claim: VC $\in NP$

Proof:

Decider: Is $\langle G, k \rangle \in VC$?

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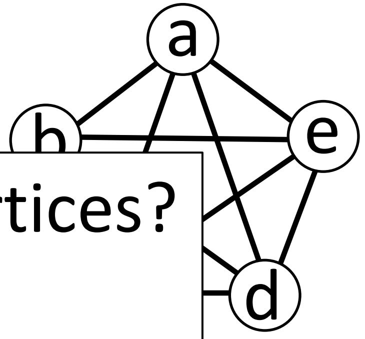

Proof:

$|V| =$

$\boldsymbol{O(1}$

$\boldsymbol{O(?}$

At most, how many edges are in an undirected graph with n vertices?

What graph has the most number of edges?

Complete graph (every pair of vertices have an edge).

How many edges does a complete graph with n vertices have?

How many edges leave each vertex? $n - 1$

How much does that all add up to? $n(n-1)$

Did we double count any edges? Yes

So how many edges are there? $\frac{n(n-1)}{2} \in O(n^2)$

What if it is directed? $n(n-1) \in O(n^2)$ (no double counts)

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$\boldsymbol{O(n)}$ ⟶ 2.1 Test if $a \in V'$ or $b \in V'$, reject if neither.

$\boldsymbol{O(1)}$ ⟶ 3. accept.

Vertex Cover (VC)

Vertex Cover: Given graph $G = (V, E)$ and integer $k \leq |V|$, is there $V' \subseteq V$, with $|V'| \leq k$, such that each edge in E contains an end point in V'?

Claim: VC $\in NP$

Proof:

Build a polynomial time verifier.

$|V| = n.$

M = on input $\langle\langle G, k\rangle, V'\rangle$, where V' is a subset of V.

$\boldsymbol{O(1)} \longrightarrow$ 1. Test if $|V'| \leq k$, reject if not.

$\boldsymbol{O(n^2)} \longrightarrow$ 2. For each edge $e = (a, b)$ in E,

$\boldsymbol{O(n)} \longrightarrow$ 2.1 Test if $a \in V'$ or $b \in V'$, reject if neither.

$\boldsymbol{O(1)} \longrightarrow$ 3. accept.

For $|V| = n$, M runs in $O(n^3)$ time, therefore $VC \in NP$.