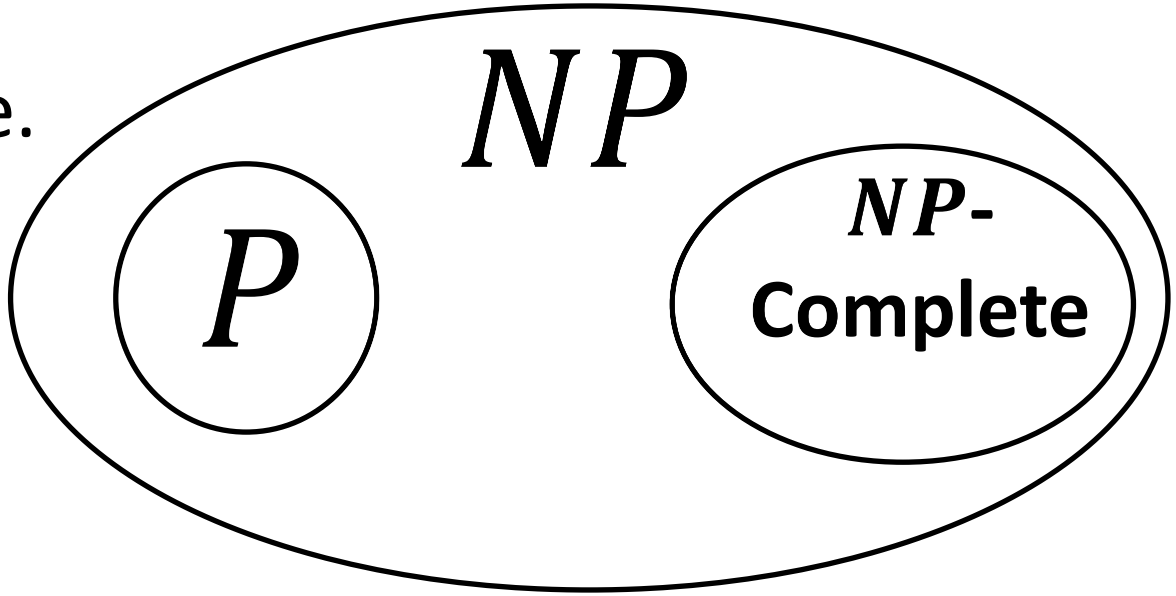
NP-Complete
CSCI 338

NP-Complete

P: Set of problems we can solve in polynomial time.

NP: Set of problems we can verify (non-deterministically solve) in polynomial time.



NP-Complete: Set of problems in NP whose solutions can solve everything in NP in polynomial extra time.

NP-Complete

How to show something (B) is in NP-Complete?

B is in NP-Complete if it satisfies two conditions:

1. $B \in NP$.
2. For some $A \in NP$-C, $A \leq_P B$.

NP-Complete

How to show something (B) is in NP-Complete?

1. Show it is in NP.

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2. Pick some known NP-Complete problem A.

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1. Show it is in NP.
2. Pick some known NP-Complete problem A.
3. Show that a solver for B can solve A in polynomial extra time.

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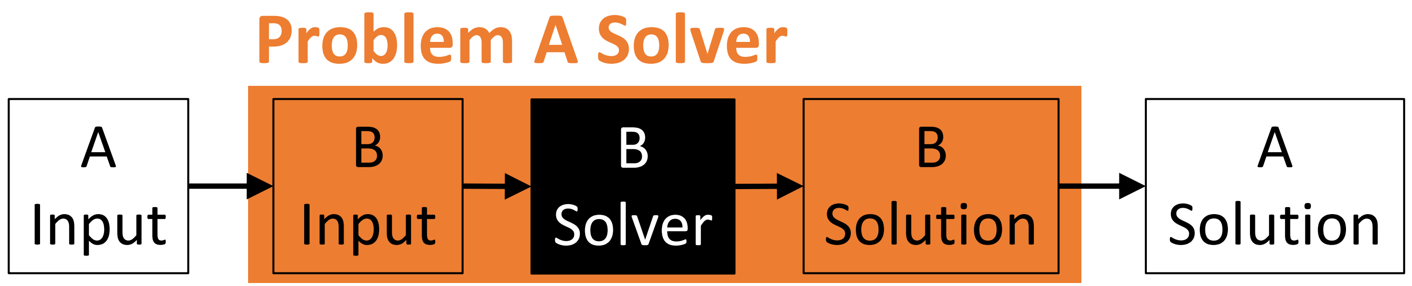

A reduces to B if A can be solved with a solver for B.

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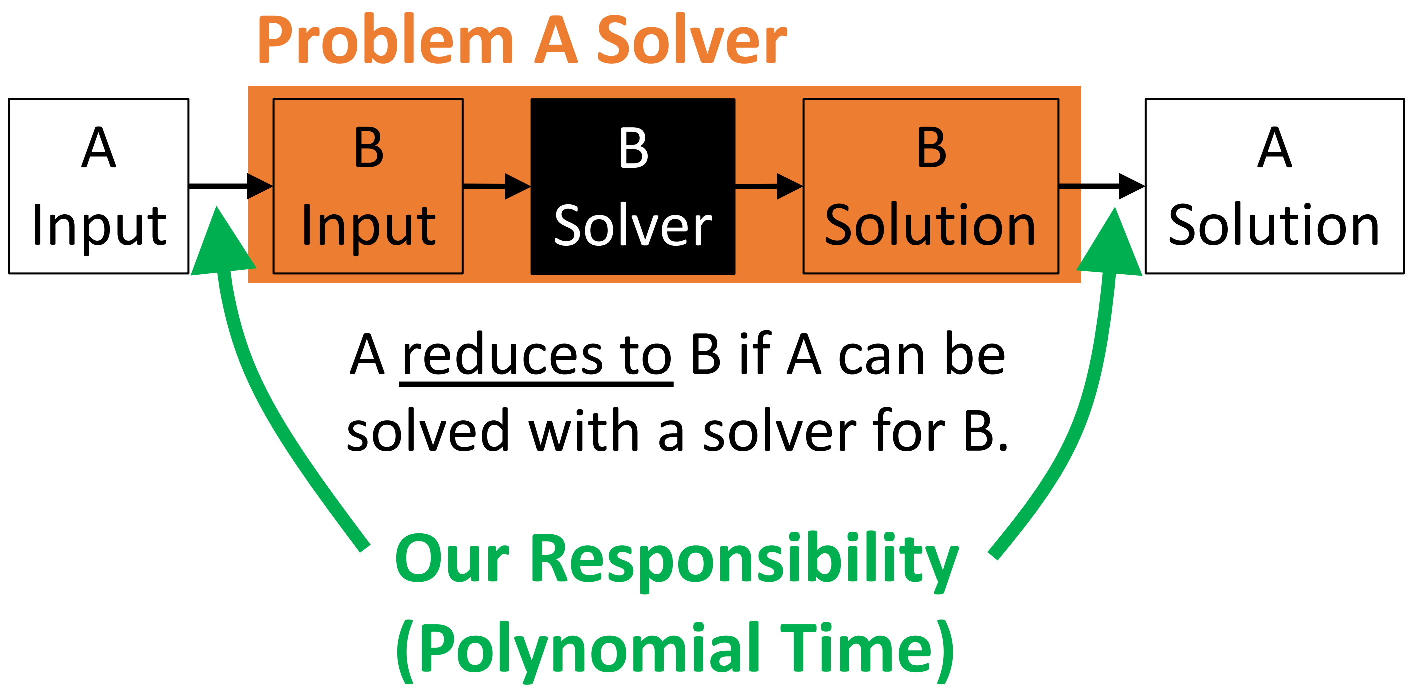

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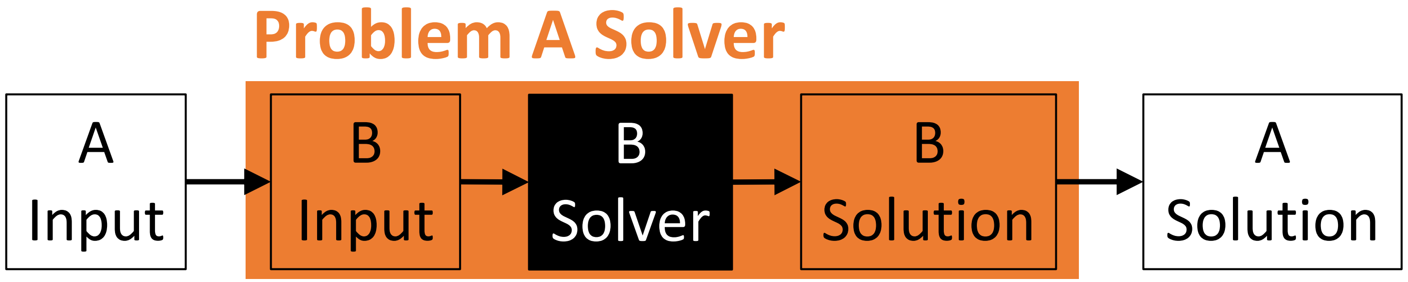

To show A reduces to B:

- Show ***every*** instance of A can be translated to ***some*** instance of B.
- The solution to B can be translated back to a solution to A.

B is in NP-Complete if it satisfies two conditions:

1. $B \in NP$.
2. For some $A \in NP$-C, $A \leq_P B$.

$3SAT$

Claim: $3SAT$ is in NP-Complete.

Proof:

$3SAT$

Claim: $3SAT$ is in NP-Complete.

Proof:

$$3SAT = \{\langle \phi \rangle : \phi \text{ is a satisfiable formula with 3 variables per clause}\}$$

$$\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$(F \vee F \vee T) \qquad (T \vee F \vee F) \qquad (T \vee T \vee T)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$x_1 = F$

$x_2 = T$

$$T \qquad\qquad T \qquad\qquad T$$

$$\searrow \quad \downarrow \quad \swarrow$$

$$T$$

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B is in NP-Complete if it satisfies two conditions:

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Claim: $3SAT$ is in NP-Complete.

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1. $B \in NP$.
2. For some $A \in NP$-C, $A \leq_P B$.

$3SAT$

Claim: $3SAT$ is in NP-Complete.

Proof:

1. Show $3SAT$ is in NP.

   Given the Boolean formula and variable assignments, evaluate the formula and accept if true and reject if false. This can be done in $O(n)$ time where n is the number of clauses.

B is in NP-Complete if it satisfies two conditions:

1. $B \in NP$.
2. For some $A \in NP$-C, $A \leq_P B$.

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Claim: $3SAT$ is in NP-Complete.

Proof:

1. Show $3SAT$ is in NP.

2. Show some NP-C problem can be solved using an algorithm for $3SAT$.

B is in NP-Complete if it satisfies two conditions:

1. $B \in NP$.
2. For some $A \in NP$-C, $A \leq_P B$.

$3SAT$

Claim: $3SAT$ is in NP-Complete.

Proof:

1. Show $3SAT$ is in NP.

2. Show ~~some NP-C problem~~ **SAT** can be solved using an algorithm for $3SAT$.

B is in NP-Complete if it satisfies two conditions:

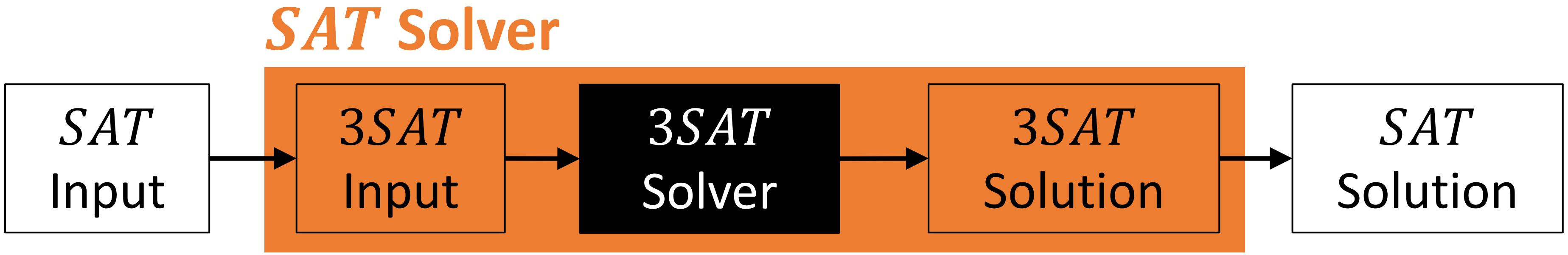
1. $B \in NP$.
2. For some $A \in NP$-C, $A \leq_P B$.

$SAT \leq_P 3SAT$

SAT is reducible to $3SAT$ in polynomial time.

Claim: $SAT \leq_P 3SAT$

Proof:



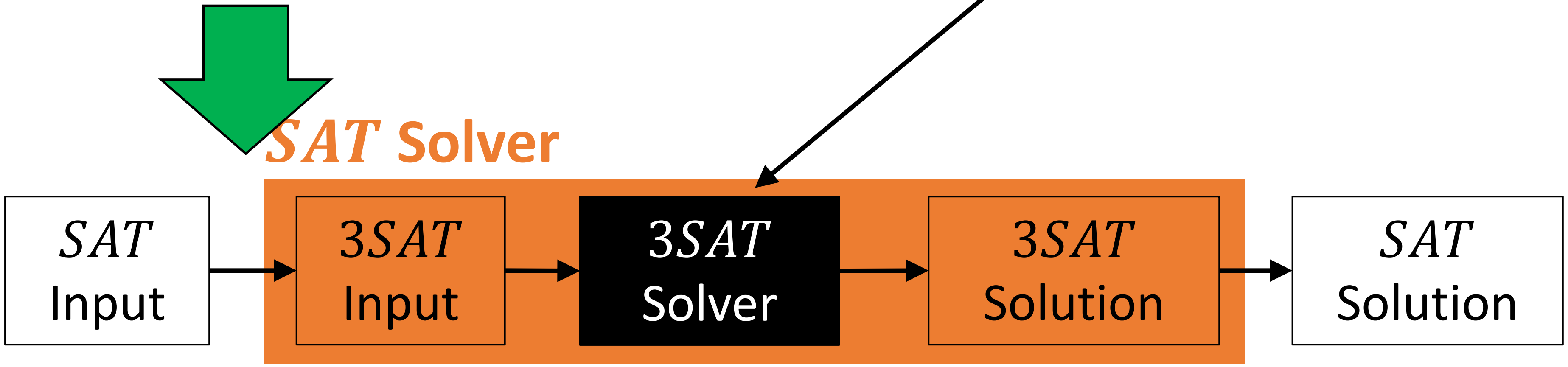
$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

So we can use our $3SAT$ solver.

Proof:

We need to turn instances of SAT into instances of $3SAT$.



$SAT \leq_P 3SAT$

$$\boldsymbol{\phi} = (\boldsymbol{x_1}) \wedge (\overline{\boldsymbol{x_1}} \vee \overline{\boldsymbol{x_2}} \vee \overline{\boldsymbol{x_2}} \vee \boldsymbol{x_1}) \wedge (\overline{\boldsymbol{x_1}} \vee \boldsymbol{x_2})$$

vs

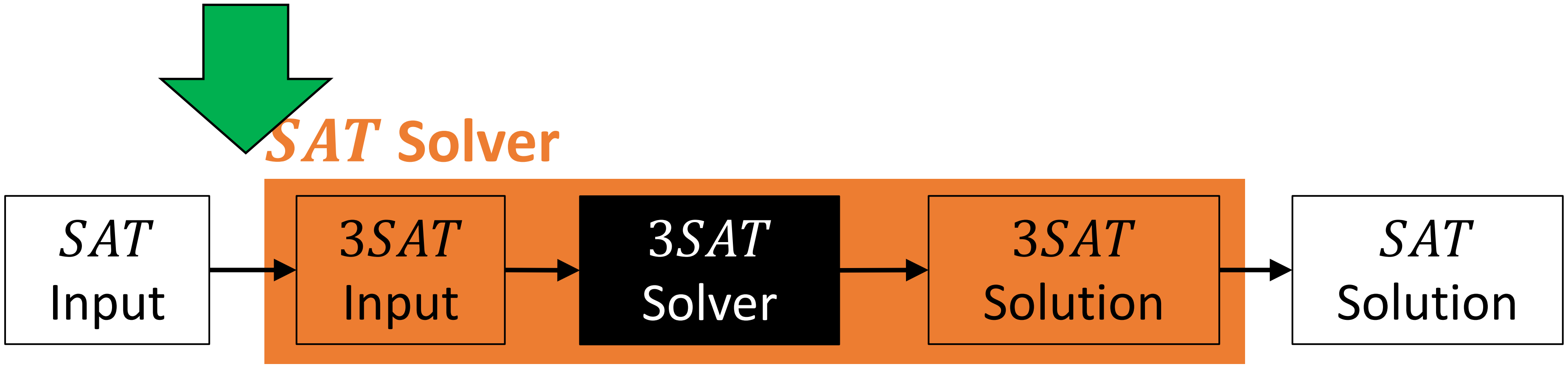
$$\boldsymbol{\phi} = (\boldsymbol{x_1} \vee \boldsymbol{x_1} \vee \boldsymbol{x_2}) \wedge (\overline{\boldsymbol{x_1}} \vee \overline{\boldsymbol{x_2}} \vee \overline{\boldsymbol{x_2}}) \wedge (\overline{\boldsymbol{x_1}} \vee \boldsymbol{x_2} \vee \boldsymbol{x_2})$$

Claim: $SAT \leq_P 3SAT$

Proof:

We need to turn instances of SAT into instances of $3SAT$.

What is keeping our SAT instance from being a $3SAT$ instance?



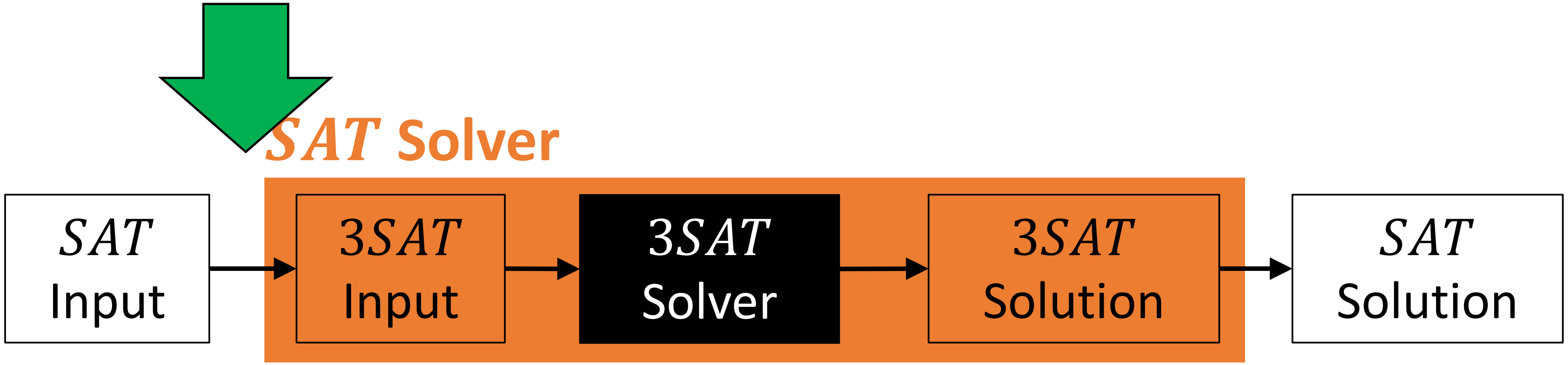
$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof:

We need to turn instances of SAT into instances of $3SAT$.

If a clause has one literal?



$SAT \leq_P 3SAT$

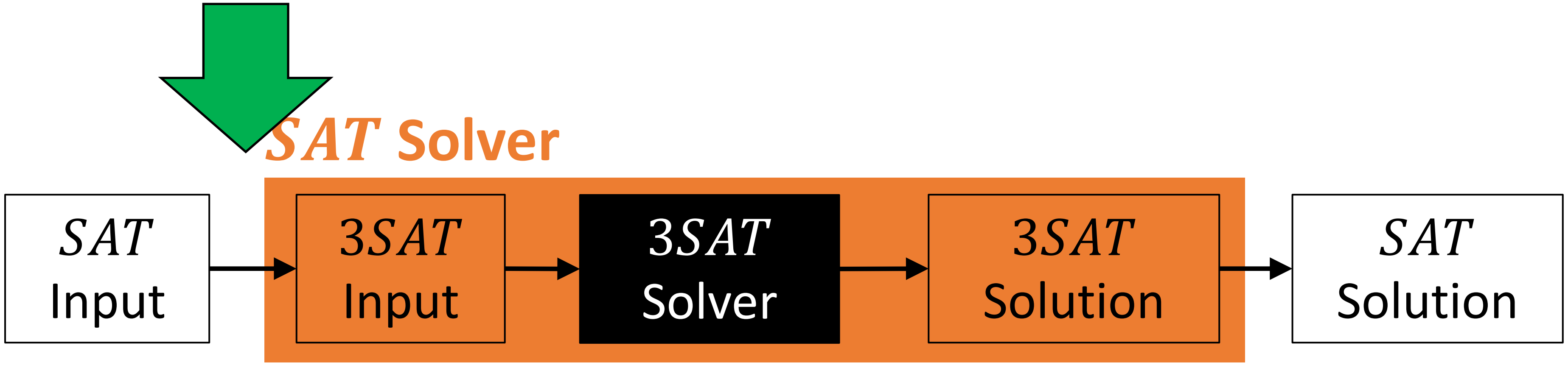
Claim: $SAT \leq_P 3SAT$

Proof:

We need to turn instances of SAT into instances of $3SAT$.

If a clause has one literal? $(x_1) \rightarrow (x_1 \vee x_1 \vee x_1)$

If a clause has two literals?



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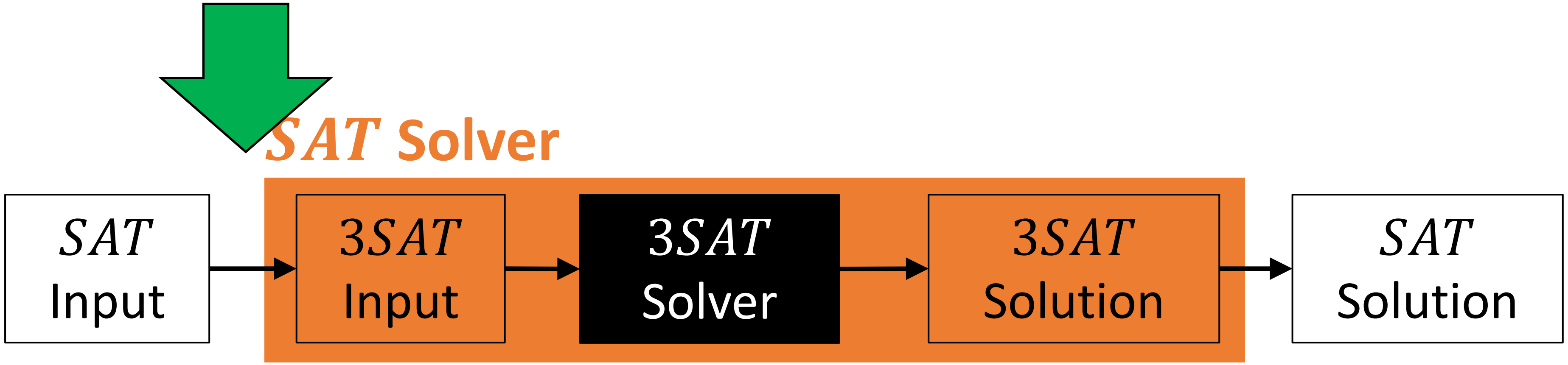
We need to turn instances of SAT into instances of $3SAT$.

If a clause has one literal? $(x_1) \rightarrow (x_1 \vee x_1 \vee x_1)$

If a clause has two literals? $(x_1 \vee x_2) \rightarrow (x_1 \vee x_1 \vee x_2)$

If a clause had three literals?



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Claim: $SAT \leq_P 3SAT$

Proof:

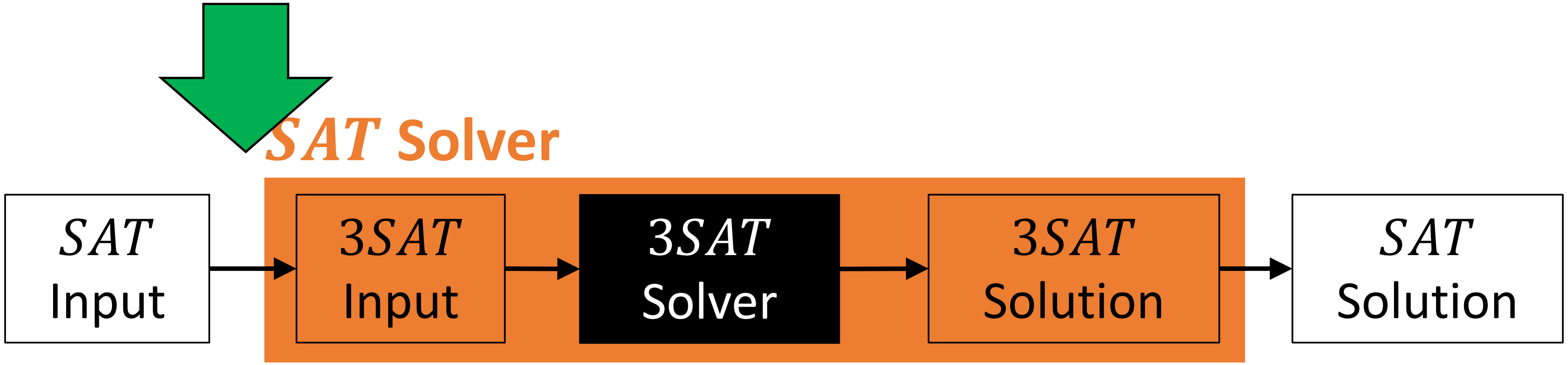
We need to turn instances of SAT into instances of $3SAT$.

If a clause has one literal? $(x_1) \rightarrow (x_1 \vee x_1 \vee x_1)$

If a clause has two literals? $(x_1 \vee x_2) \rightarrow (x_1 \vee x_1 \vee x_2)$

If a clause had three literals? No change.

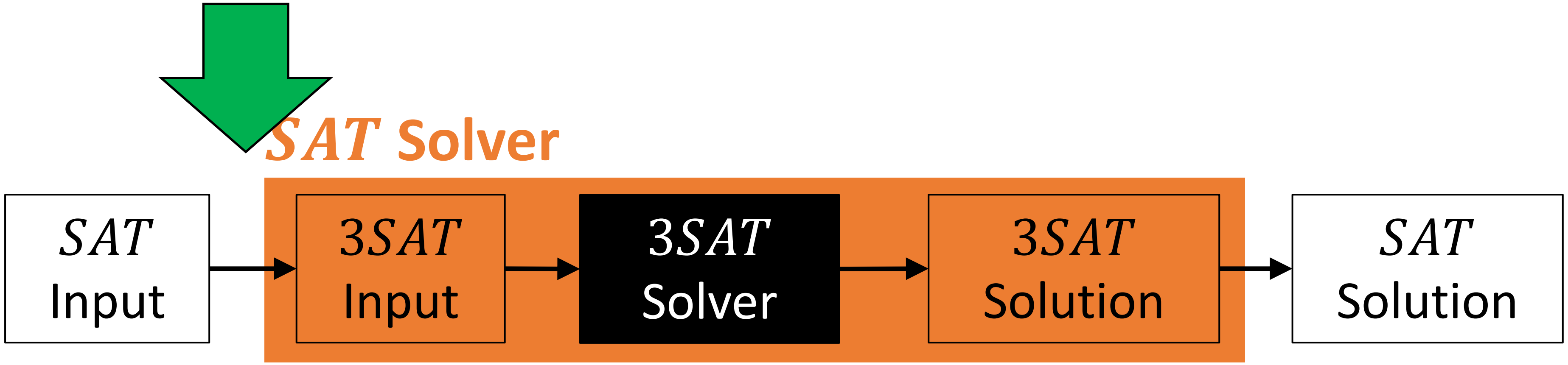
If a clause has more than three literals?



$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with >3 literals into $3SAT$ clauses.

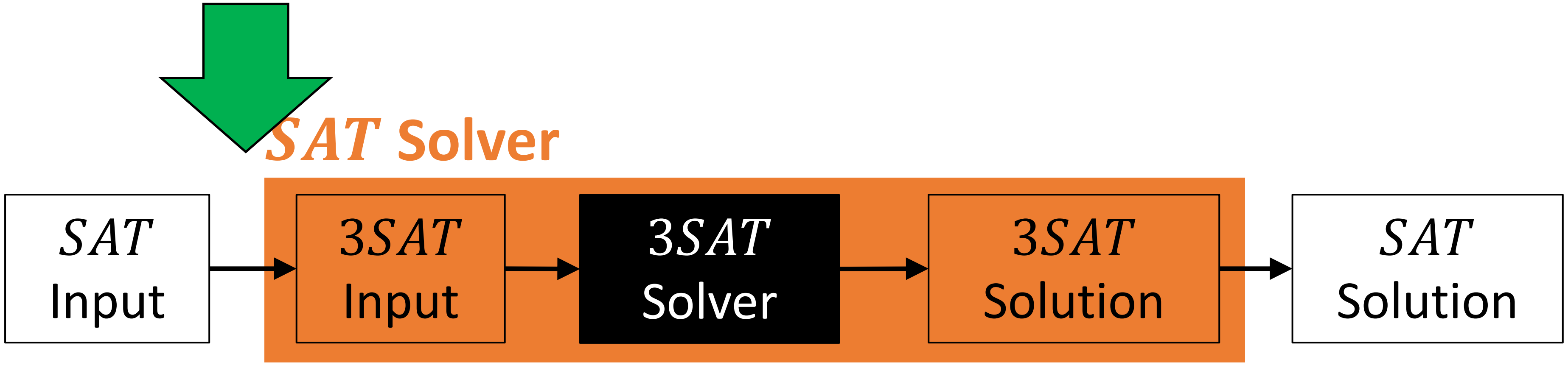


$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with >3 literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_k)$$



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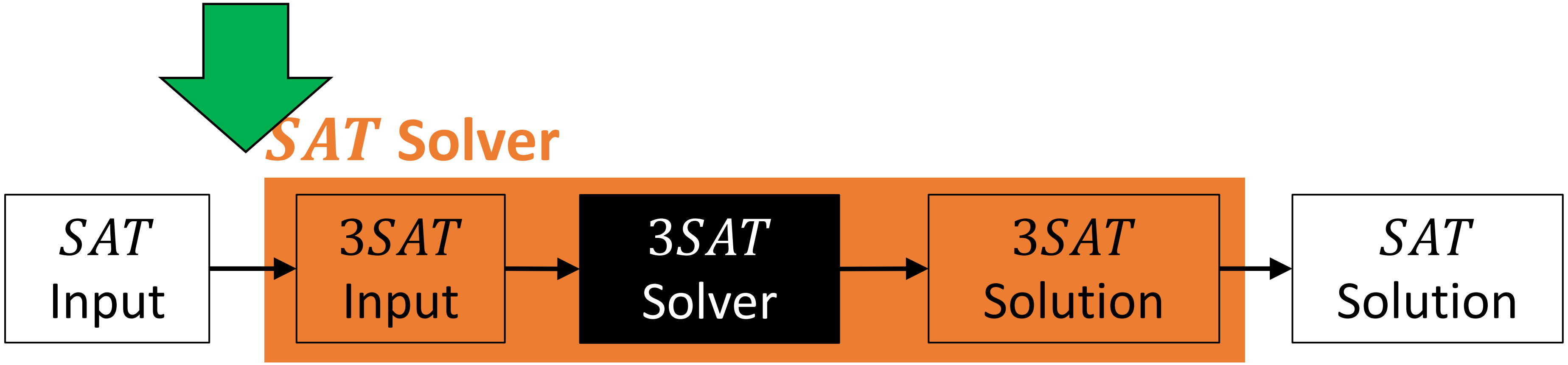
$$\phi_{SAT} = (x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_k)$$
$$\rightarrow \phi_{3SAT} = ?$$



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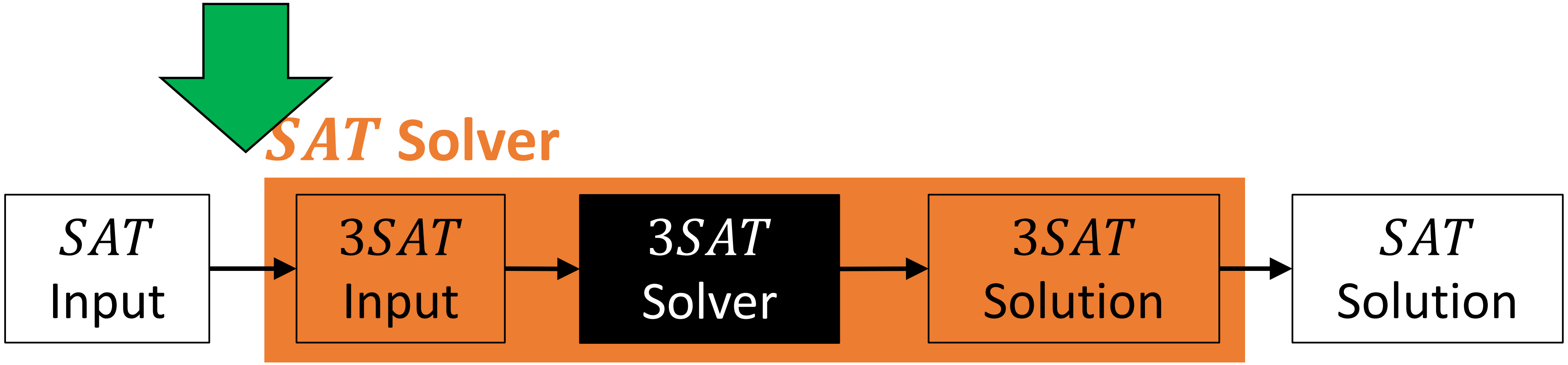
$$\rightarrow \phi_{3SAT} = (x_1 \vee x_2 \vee \mathbf{z_1}) \wedge (\overline{\mathbf{z_1}} \vee x_3 \vee \mathbf{z_2}) \wedge \cdots \wedge (\overline{\mathbf{z_{k-3}}} \vee x_{k-1} \vee x_k)$$



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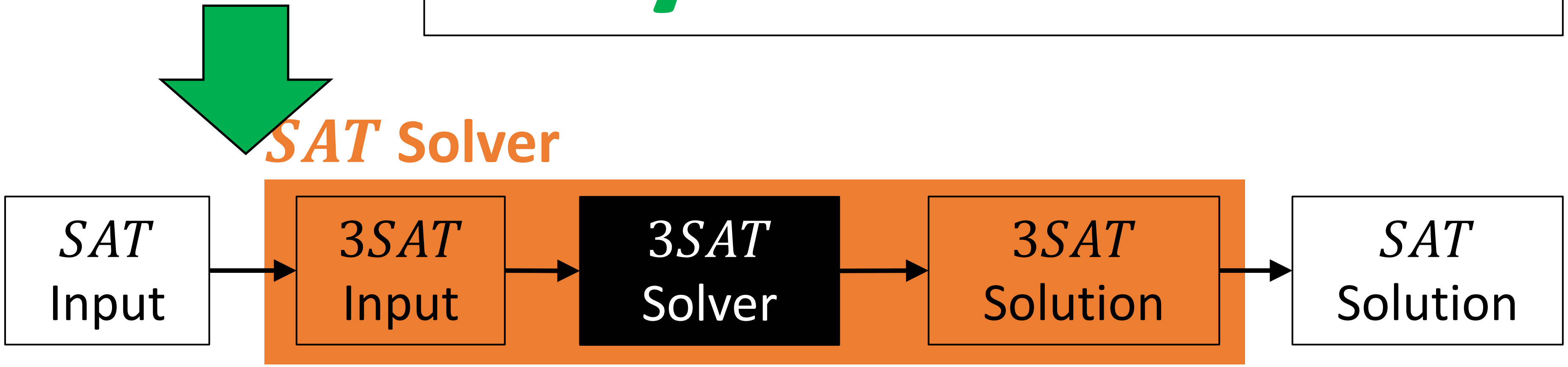
Polynomial Time?



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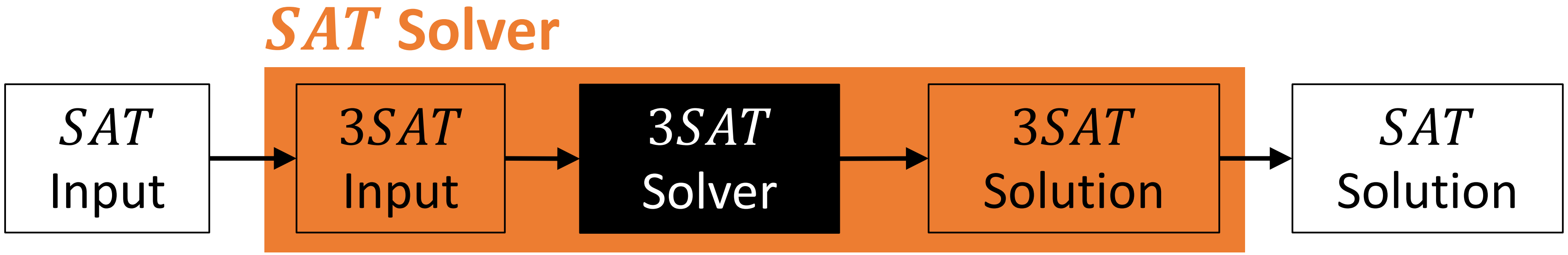
Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.

We need to show that if the 3SAT Solver says the 3SAT input is satisfiable, the SAT input is too AND if the 3SAT input is not, the SAT input isn't either.



$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with >3 literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \lor x_2 \lor x_3 \lor \cdots \lor x_k)$$

$$\rightarrow \phi_{3SAT} = (x_1 \lor x_2 \lor \mathbf{z_1}) \land (\overline{\mathbf{z_1}} \lor x_3 \lor \mathbf{z_2}) \land \cdots \land (\overline{\mathbf{z_{k-3}}} \lor x_{k-1} \lor x_k)$$

Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.

Suppose ϕ_{SAT} can be true.

$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with >3 literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_k)$$
$$\rightarrow \phi_{3SAT} = (x_1 \vee x_2 \vee \boldsymbol{z_1}) \wedge (\overline{\boldsymbol{z_1}} \vee x_3 \vee \boldsymbol{z_2}) \wedge \cdots \wedge (\overline{\boldsymbol{z_{k-3}}} \vee x_{k-1} \vee x_k)$$

Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.

Suppose ϕ_{SAT} can be true. Then some x_m is true.

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$$\ldots \wedge (\overline{z_{i-1}} \vee x_{m-1} \vee z_i) \wedge (\overline{z_i} \vee x_m \vee z_{i+1}) \wedge (\overline{z_{i+1}} \vee x_{m+1} \vee z_{i+2}) \wedge \ldots$$

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Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with >3 literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_k)$$

$$\rightarrow \phi_{3SAT} = (x_1 \vee x_2 \vee \mathbf{z_1}) \wedge (\overline{\mathbf{z_1}} \vee x_3 \vee \mathbf{z_2}) \wedge \cdots \wedge (\overline{\mathbf{z_{k-3}}} \vee x_{k-1} \vee x_k)$$

Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.

Suppose ϕ_{SAT} can be true. Then some x_m is true. Let x_m be true in ϕ_{3SAT}.

$$\ldots \wedge (\overline{z_{i-1}} \vee x_{m-1} \vee z_i) \wedge (\overline{z_i} \vee x_m \vee z_{i+1}) \wedge (\overline{z_{i+1}} \vee x_{m+1} \vee z_{i+2}) \wedge \ldots$$

$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with > 3 literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_k)$$

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Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.

Suppose ϕ_{SAT} can be true. Then some x_m is true. Let x_m be true in ϕ_{3SAT}. Let all z_i's before x_m be true…

$$\ldots \wedge (\overline{z_{i-1}} \vee x_{m-1} \vee z_i) \wedge (\overline{z_i} \vee x_m \vee z_{i+1}) \wedge (\overline{z_{i+1}} \vee x_{m+1} \vee z_{i+2}) \wedge \ldots$$

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Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with > 3 literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_k)$$
$$\rightarrow \phi_{3SAT} = (x_1 \vee x_2 \vee \mathbf{z_1}) \wedge (\overline{\mathbf{z_1}} \vee x_3 \vee \mathbf{z_2}) \wedge \cdots \wedge (\overline{\mathbf{z_{k-3}}} \vee x_{k-1} \vee x_k)$$

Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.

Suppose ϕ_{SAT} can be true. Then some x_m is true. Let x_m be true in ϕ_{3SAT}. Let all z_i's before x_m be true and all z_i's after be false.

$$\ldots \wedge (\overline{z_{i-1}} \vee x_{m-1} \vee z_i) \wedge (\overline{z_i} \vee x_m \vee z_{i+1}) \wedge (\overline{z_{i+1}} \vee x_{m+1} \vee z_{i+2}) \wedge \ldots$$

$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with > 3 literals into $3SAT$ clauses.

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Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.

Suppose ϕ_{SAT} can be true. Then some x_m is true. Let x_m be true in ϕ_{3SAT}. Let all z_i's before x_m be true and all z_i's after be false.

$$\ldots \wedge (\overline{z_{i-1}} \vee x_{m-1} \vee z_i) \wedge (\overline{z_i} \vee x_m \vee z_{i+1}) \wedge (\overline{z_{i+1}} \vee x_{m+1} \vee z_{i+2}) \wedge \ldots$$

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Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with >3 literals into $3SAT$ clauses.

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Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.

Suppose ϕ_{SAT} can be true. Then some x_m is true. Let x_m be true in ϕ_{3SAT}. Let all z_i's before x_m be true and all z_i's after be false.
$\Rightarrow$ Every clause has a variable set to true.

$SAT \leq_P 3SAT$

Claim: $SAT \leq_P 3SAT$

Proof: Convert SAT clauses with $>$ 3 literals into $3SAT$ clauses.

$$\phi_{SAT} = (x_1 \vee x_2 \vee x_3 \vee \cdots \vee x_k)$$

$$\rightarrow \phi_{3SAT} = (x_1 \vee x_2 \vee \boldsymbol{z_1}) \wedge (\overline{\boldsymbol{z_1}} \vee x_3 \vee \boldsymbol{z_2}) \wedge \cdots \wedge (\overline{\boldsymbol{z_{k-3}}} \vee x_{k-1} \vee x_k)$$

Need to show: ϕ_{SAT} can be true $\Leftrightarrow$ ϕ_{3SAT} can be true.

Suppose ϕ_{SAT} can be true. Then some x_m is true. Let x_m be true in ϕ_{3SAT}. Let all z_i's before x_m be true and all z_i's after be false.
$\Rightarrow$ Every clause has a variable set to true. $\therefore \phi_{3SAT} = T$.

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$\Rightarrow$ Every clause has a variable set to true. $\therefore \phi_{3SAT} = T$.

Suppose ϕ_{3SAT} can be true.